Math 5B - HW3 (Written Portion)

Due Aug 21, in my mailbox in SH 6623

You must show your work to receive credit.

For problems 1-3, find all critical points (if any) of the given function $f(x,y)$, and determine whether they are local extreme points or saddle points.

1. $f(x,y) = xy + \frac{x+y}{xy}$.

   Since $\nabla f(x,y) = (y - \frac{1}{x^2}, x - \frac{1}{y^2})$, we have that

   $$\nabla f(x,y) = (0,0) \;\Rightarrow\; y - \frac{1}{x^2} = 0 \quad x - \frac{1}{y^2} = 0.$$

   Solving for y in the first equation and substituting into the second gives $x - x^4 = 0 \Rightarrow x(1-x^3) = 0 \Rightarrow x = 0$ or $x = 1$. $x = 0$ is not possible by the first equation, so $x = 1$. Then $y = 1$, so the only critical point is $(1,1)$. To determine whether it is a extreme point or a saddle point:

   $$Hf(x,y) = \begin{bmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{bmatrix}.$$

   Since $|Hf(1,1)| = 3$ and $f_{xx}(1,1) = 2$, we see that the critical point is a local minimum.

2. $f(x,y) = xye^{-x^2-y^2}$.

   First, find the critical points:

   $$\nabla f(x,y) = e^{-x^2-y^2}(y - 2x^2y, x - 2xy^2).$$

   Setting $\nabla f(x,y) = (0,0)$ gives 5 critical points:

   $$(0,0),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

   (See lecture notes from 8/20 to see how these were computed). The Hessian matrix:

   $$Hf(x,y) = e^{-x^2-y^2}\begin{bmatrix} 2xy(2x^2-3) & (2x^2-1)(2y^2-1) \\ (2x^2-1)(2y^2-1) & 2xy(2y^2-3) \end{bmatrix}.$$

   Calculating determinants:

   $$|Hf(1/\sqrt{2}, 1/\sqrt{2})| = 4e^{-2}, \qquad |Hf(1/\sqrt{2}, -1/\sqrt{2})| = 4e^{-2},$$
   $$|Hf(-1/\sqrt{2}, 1/\sqrt{2})| = 4e^{-2}, \qquad |Hf(-1/\sqrt{2}, -1/\sqrt{2})| = 4e^{-2},$$
   $$|Hf(0,0)| = -1.$$

   We see that $(0,0)$ is a saddle point, and since

   $$f_{xx}(1/\sqrt{2}, 1/\sqrt{2}) = -2e^{-1}, \qquad f_{xx}(1/\sqrt{2}, -1/\sqrt{2}) = 2e^{-1},$$
   $$f_{xx}(-1/\sqrt{2}, 1/\sqrt{2}) = 2e^{-1}, \qquad f_{xx}(-1/\sqrt{2}, -1/\sqrt{2}) = -2e^{-1},$$

   we see that $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$ are local maximums, and that $(-1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$ are local minimums.

3. $f(x, y) = x^3 + y^3 + 3x^2y - 3y$.

   Calculating the critical points:

   $$\nabla f(x, y) = (3x^2 + 6xy, 3y^2 + 3x^2 - 3).$$

   Setting $\nabla f(x, y) = (0, 0)$ gives $3x(x + 2y) = 0$, so either $x = 0$ or $x = -2y$. If $x = 0$, then $3y^2 + 3(0)^2 - 3 = 0 \Rightarrow y = \pm 1$, so $(0, 1)$, $(0, -1)$ are critical points. If $x = -2y$, then $3y^2 + 3(-2y)^2 - 3 = 0 \Rightarrow y = \pm \frac{1}{\sqrt{5}}$. So there are 4 total critical points:

   $$(0, 1), \quad (0, -1) \quad (-2/\sqrt{5}, 1/\sqrt{5}), \quad (2/\sqrt{5}, -1/\sqrt{5}).$$

   Calculating the Hessian matrix:

   $$Hf(x, y) = 6 \begin{bmatrix} x + y & x \\ x & y \end{bmatrix}.$$

   Computing determinants:

   $$|Hf(0, 1)| = 36, \qquad |Hf(0, -1)| = 36,$$
   $$|Hf(-2/\sqrt{5}, 1/\sqrt{5})| = -6, \qquad |Hf(2/\sqrt{5}, -1/\sqrt{5})| = -6.$$

   We see that $(-2/\sqrt{5}, 1/\sqrt{5})$, $(2/\sqrt{5}, -1/\sqrt{5})$ are both saddle points, and since

   $$f_{xx}(0, 1) = 6, \quad f_{xx}(0, -1) = -6,$$

   we see that $(0, 1)$ is local minimum, and that $(0, -1)$ is a local maximum.

For problems 4-6, find the extreme values (if any) of a function f subject to the given constraint.

4. $f(x, y) = 3xy, \qquad x^2 + y^2 = 4.$

   First we compute gradients:

   $$\nabla f(x, y) = (3y, 3x), \quad \nabla g(x, y) = (2x, 2y).$$

   Using Lagrange multipliers, we set $\nabla f = \lambda \nabla g$ to get

   $$3y = 2\lambda x, \quad 3x = 2\lambda y.$$

   From the first equation we get that $\lambda = \frac{3y}{2x}$, and substituting this into the second equation gives $3x^2 = 3y^2 \Rightarrow x = \pm y$. Using the constraint equation, we find that there are four critical points:

   $$(\sqrt{2}, \sqrt{2}), \ (-\sqrt{2}, \sqrt{2}), \ (\sqrt{2}, -\sqrt{2}), \ (-\sqrt{2}, -\sqrt{2}).$$

   Plugging these points into f, we see that

   $$f(\sqrt{2}, \sqrt{2}) = 6 \qquad f(-\sqrt{2}, \sqrt{2}) = -6$$
   $$f(\sqrt{2}, -\sqrt{2}) = -6 \qquad f(-\sqrt{2}, -\sqrt{2}) = 6,$$

   so the maximum of $f(x, y)$ constrained to $g(x, y) = 4$ is 6, and the minimum is -6.

5. $f(x, y) = 2x^2 - y^2, \qquad x^2 + y^2 = 1.$

   First, we calculate the gradients:

   $$\nabla f(x, y) = (4x, -2y), \quad \nabla g(x, y) = (2x, 2y).$$

   Setting $\nabla f = \lambda \nabla g$ gives

   $$4x = 2\lambda x, \quad -2y = 2\lambda y.$$

   From the first equation, we see that either $\lambda = 2$ or $x = 0$. If $x = 0$, then by the constraint equation $y = \pm 1$. If $\lambda = 2$, we substitute this into the second equation to get $y = 0$; using the constraint equation gives $x = \pm 1$ in this case. Thus there are 4 critical points:

   $$(1, 0), \ (-1, 0), \ (0, 1), \ (0, -1).$$

   Plugging these points into f:

   $$f(1, 0) = 2, \quad f(-1, 0) = 2, \quad f(0, 1) = -1, \quad f(0, -1) = -1,$$

   we see that the maximum of f constrained to $g(x, y) = 1$ is 2, and the minimum is -1.

6. $f(x, y, z) = xyz, \qquad x^2 + y^2 + z^2 = 9.$

   Computing gradients:

   $$\nabla f(x, y, z) = (yz, xz, xy), \quad \nabla g(x, y, z) = (2x, 2y, 2z).$$

   Setting $\nabla f = \lambda \nabla g$ gives

   $$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z.$$

   The first equation says that $\lambda = \frac{yz}{2x}$; substituting into the second equation gives $x^2 = y^2$, and substituting into the third equation gives $x^2 = z^2$. Substituting these identities into the constraint equation gives $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$. Then this gives 8 (yeesh!) critical points:

   $$(\sqrt{3}, \sqrt{3}, \sqrt{3}), (-\sqrt{3}, \sqrt{3}, \sqrt{3}), (\sqrt{3}, -\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3}, \sqrt{3}),$$
   $$(\sqrt{3}, \sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}, -\sqrt{3}), (\sqrt{3}, -\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, -\sqrt{3}, -\sqrt{3}).$$

   Substituting all these points into f, we see that the maximum of $f(x, y, z)$ constrained to $g(x, y, z) = 9$ is $3\sqrt{3}$, and the minimum is $3\sqrt{3}$.

7. Using the method of Lagrange multipliers, find the minimum distance from the surface $x^2 + y^2 - z^2 = 4$ to the origin.

The distance from the origin function is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. However, minimized this function is equivalent to minimizing $d^2(x, y, z) = x^2 + y^2 + z^2$. Computing some gradients:

$$\nabla d^2(x, y, z) = (2x, 2y, 2z), \quad \nabla g(x, y, z) = (2x, 2y, -2z).$$

Setting $\nabla d^2 = \lambda \nabla g$ gives

$$2x = 2\lambda x, \quad 2y = 2\lambda y, \quad 2z = -2\lambda z.$$

The third equation says $2z(1+\lambda) = 0$, so either $\lambda = -1$, or $z = 0$. If $\lambda = -1$, then the first two equations say that $x = y = 0$, which leaves the constraint equation saying $-z^2 = 4$, which is impossible, so it must be the case that $z = 0$. In this case, we have that every point of the form (x, y) such that $x^2 + y^2 = 4$ is a critical point. For each of these points, $d^2(x, y, z) = 4$, so the minimum distance is 2.

8. Consider the partial differential equation $u_t = cu_x$, where c is some constant. Verify that under the change of variables

$$v = x + ct \qquad\qquad w = t,$$

u satisfies $u_w = 0$.

First, we write u_t and u_x in terms of u_v and u_w:

$$\begin{aligned} u_t &= u_v v_t + u_w w_t \\ &= c u_v + u_w, \\ u_x &= u_v v_x + u_w w_x \\ &= u_v. \end{aligned}$$

Substituting these into the differential equation gives

$$cu_v + u_w = c(u_v) \quad \Rightarrow \quad u_w = 0.$$