

A UNIQUENESS CRITERION IN THE MULTIVARIATE MOMENT PROBLEM

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ABSTRACT. A determinacy criterion for the multivariate Hamburger moment problem is derived from a recent existence by extension result, [10].

1. INTRODUCTION

The present note is a companion to the article [10]. By exploiting an existence result of [10] we derive below a uniqueness criterion for Hamburger's moment problem in any number of dimensions. Typically, the known determinacy criteria are stated in terms of density of polynomials in certain weighted L^p norms, cf. [3], [7], [12]. A notable exception is the Carleman type condition of [5]. We propose below a numerical sufficient condition of determinacy, completely expressible in terms of some associated orthogonal polynomials. We follow the path via a variational problem first studied by M. Riesz, [11].

First let us fix some notation. Let d be a positive integer and let $x = (x_1, x_2, \dots, x_d)$ be the coordinates in \mathbf{R}^d . When embedding (naturally) \mathbf{R}^d into \mathbf{C}^d we will denote by $z = (z_1, z_2, \dots, z_d)$ the complex coordinates. We put $z \cdot z = z_1^2 + z_2^2 + \dots + z_d^2$, so that the euclidean norm of the vector x is $|x| = \sqrt{x \cdot x}$. The algebra of polynomials in the indeterminates x will be denoted by $\mathbf{R}[x]$, in the case of real coefficients, and by $\mathbf{C}[z]$ when allowing complex coefficients. For a fixed positive integer n , the space of polynomials of degree less or equal than n will be denoted by $\mathbf{R}_n[x]$, respectively $\mathbf{C}_n[z]$. Whenever it will be necessary, the domain of the polynomial map associated to an element $p \in \mathbf{R}[x]$ will automatically be extended to \mathbf{C}^d . Throughout this note we denote $\mathbf{N} = \{0, 1, 2, 3, \dots\}$.

Let μ be a positive, rapidly decreasing at infinity measure on \mathbf{R}^d , and let $\mathbf{a} = (a_\alpha)_{\alpha \in \mathbf{N}^d}$ be the corresponding moment sequence:

$$a_\alpha = \int_{\mathbf{R}^d} x^\alpha d\mu(x), \quad \alpha \in \mathbf{N}^d.$$

Partially supported by the National Science Foundation Grant DMS 9800666.

Associated solely to the moment sequence is the integration functional:

$$L(p) = \int_{\mathbf{R}^d} p d\mu, \quad p \in \mathbf{R}[x].$$

First we recall some basic facts in dimension $d = 1$. Let n be a positive integer, and let us consider (after Riesz [11]) the variational problem:

$$\rho_n = \min\{L(p^2); p \in \mathbf{R}_n[x], |p(\pm i)| = 1\}. \quad (1)$$

The sequence ρ_n is obviously decreasing and the limit $\rho = \lim_{n \rightarrow \infty} \rho_n$ is equal to zero if and only if the initial moment problem is *determinate* (that is, in our notation, μ is the unique measure with moments \mathbf{a}). The real numbers ρ_n are the radii of a decreasing set of disks in the plane, representing the values (at $z = i$) of the diagonal Padé approximants of the Cauchy transform of the measure μ , see [1] for full details. Most of the uniqueness criteria in the theory of moments in one variable are related to estimates, in different terms, of the limit radius ρ .

Since the relation (1) refers to real polynomials p , we can obviously replace the condition $|p(\pm i)| = 1$ by $|p(i)| = 1$. Also, we recall that the numbers $\pm i$ are not privileged; they can be replaced by any pair $\alpha, \bar{\alpha}$ with $\alpha \notin \mathbf{R}$, see [11].

In arbitrary dimension $d \geq 1$ we can define an analogous quantity:

$$\rho_n = \min\{L(p^2); p \in \mathbf{R}_n[x], |p(z)| = 1 \text{ for } z \cdot z + 1 = 0\}, \quad (2)$$

and set $\rho = \lim_{n \rightarrow \infty} \rho_n$.

The aim of the present note is to prove that, in any dimension d , if $\rho = 0$, then the initial moment problem is determinate. We will show that actually the numbers ρ_n are computable, for instance in terms of certain orthogonal polynomials depending only on the moment sequence \mathbf{a} .

This work was completed when the first author was visiting the Mathematics Department at the University of Science and Technology of Lille. It is a pleasure for him to thank this institution for hospitality and support.

2. MAIN RESULT

Throughout this section we use the notation introduced before: \mathbf{a} is the moment sequence of the measure μ on \mathbf{R}^d , $d > 1$, with associated integration functional L defined on polynomials, and $\rho = \lim_{n \rightarrow \infty} \rho_n$, as in relation (2).

First we note that ρ_n can be interpreted as a distance in the norm $\|p\|^2 = L(p^2)$, $p \in \mathbf{R}[x]$. Indeed, the complex variety $V = \{z \in \mathbf{C}^d; z \cdot$

$z = -1\}$ is a connected smooth hypersurface in \mathbf{C}^d , $d > 1$, hence by the maximum modulus principle (cf. for instance [8] pp. 118), if a polynomial $p \in \mathbf{C}[z]$ satisfies $|p(z)| = 1, z \in V$, then there is a constant $c, |c| = 1$ such that $p(z) = c, z \in V$. To see that the variety V is connected it is sufficient to decompose a point $z \in V$ into real and imaginary parts: $z = x + iy, x, y \in \mathbf{R}^d$, and to remark that the equation of V becomes $|x|^2 - |y|^2 + 1 = 0, x \cdot y = 0$. Then, we can deform x along its direction to zero (specifically $tx, t \in [0, 1]$) and deform correspondingly y to the unit vector $y/|y|$. Thus, V is homotopically equivalent to the unit sphere in $\mathbf{R}^d, d > 1$, hence it is connected.

Moreover, a standard division argument shows that:

$$p(z) = c - (1 + z \cdot z)q(z), \quad q \in \mathbf{C}[z].$$

Indeed, the ideal generated by the polynomial $z \cdot z + 1$ is prime in every localization of the polynomial ring $\mathbf{C}[z]$, hence it is prime in $\mathbf{C}[z]$. By Hilbert Nullstellensatz ([8] pp. 404), since the polynomial $p(z) - c$ vanishes on V it can be factored by $1 + z \cdot z$.

By taking real and imaginary parts in the coefficients of q we obtain polynomials $r(x), s(x)$, such that $r(x) = \Re q(x), s(x) = \Im q(x), x \in \mathbf{R}^d$. Therefore, since we have started with a real polynomial p we obtain:

$$p(x) = \Re c - (1 + |x|^2)r(x), \quad x \in \mathbf{R}^d,$$

and

$$0 = \Im c - (1 + |x|^2)s(x), \quad x \in \mathbf{R}^d.$$

But the second condition implies $c \in \mathbf{R}$ and $s(x) = 0$, hence $c = \pm 1$. Without loss of generality we can assume henceforth that $c = 1$.

In conclusion, for $d > 1$ and $n \geq 2$ we have proved the following formula:

$$\rho_n = \min\{L(|p|^2); \quad p(z) = 1 - (1 + z \cdot z)q(z), \quad q \in \mathbf{C}_{n-2}[z]\}. \quad (3)$$

By decomposing $q(x) = r(x) + is(x), x \in \mathbf{R}^d$, as before in real and imaginary parts, we observe that:

$$|p(x)|^2 = [1 - (1 + |x|^2)r(x)]^2 + (1 + |x|^2)^2 s(x)^2, \quad x \in \mathbf{R}^d,$$

so that the minimum in the above expression of ρ_n is indeed attained on real polynomials.

Theorem 2.1. *A moment sequence with invariant $\rho = 0$ is determinate.*

Proof. As recalled before, the case $d = 1$ is classical [11], so we can assume $d > 1$, in which situation formula (3) holds. If $\rho = 0$, then there exists a sequence of polynomials $q_n \in \mathbf{R}[x]$ such that

$$\lim_{n \rightarrow \infty} L([1 - (1 + |x|^2)q_n(x)]^2) = 0.$$

Assume that ν is another positive measure, rapidly decreasing at infinity in \mathbf{R}^d and having the same moments \mathbf{a} as μ . Then we have:

$$\lim_{n \rightarrow \infty} \|1 - (1 + |x|^2)q_n(x)\|_{2,\mu} = \lim_{n \rightarrow \infty} \|1 - (1 + |x|^2)q_n(x)\|_{2,\nu} = 0.$$

Since the function $\frac{1}{1+|x|^2}$ is positive and bounded on \mathbf{R}^d , we infer:

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{1 + |x|^2} - q_n(x) \right\|_{2,\mu} = \lim_{n \rightarrow \infty} \left\| \frac{1}{1 + |x|^2} - q_n(x) \right\|_{2,\nu} = 0.$$

Let $\alpha \in \mathbf{N}^d$ be an arbitrary multi-index and let m be a non-negative integer. Our aim is to prove that:

$$\int_{\mathbf{R}^d} \frac{x^\alpha}{(1 + |x|^2)^m} d\mu(x) = \int_{\mathbf{R}^d} \frac{x^\alpha}{(1 + |x|^2)^m} d\nu(x). \quad (4)$$

Then a direct argument, or the main result of [10], can be applied and conclude that $\mu = \nu$.

We prove relation (4) by induction on $m \geq 0$. The case $m = 0$ follows from the assumption that both measures have the same moments. Assume that relation (4) is valid for m replaced by $m - 1$. Let σ be one of the measures μ, ν . Since $\frac{x^\alpha}{(1+|x|^2)^{m-1}} \in L^2(\sigma)$ and $q_n(x) \rightarrow \frac{1}{1+|x|^2}$ in $L^2(\sigma)$, we obtain $\frac{x^\alpha q_n(x)}{(1+|x|^2)^{m-1}} \rightarrow \frac{x^\alpha}{(1+|x|^2)^m}$ in $L^1(\sigma)$. But according to the induction hypothesis this implies (4).

□

Note that in the above proof only the convergence $\|q_n(x) - \frac{1}{1+|x|^2}\|_{2,\mu} \rightarrow 0$ was used. However, this latter condition is not intrinsic in the moments \mathbf{a} .

Since, by formula (3), $\sqrt{\rho}$ is the distance in $L^2(\mu)$ between the constant function $\mathbf{1}$ and the subspace $(1 + |x|^2)\mathbf{C}[z]$, we obtain the following constructive way of computing this number.

Corollary 2.2. *Let $P_\alpha(x)$, $\alpha \in \mathbf{N}^d$, be a sequence of orthonormal polynomials with respect to the measure $(1 + |x|^2)^2 d\mu(x)$ and define the coefficients:*

$$c_\alpha = \int_{\mathbf{R}^d} P_\alpha(x)(1 + |x|^2) d\mu(x), \quad \alpha \in \mathbf{N}^d. \quad (5)$$

Then

$$\rho = a_0^2 - \sum_{\alpha \in \mathbf{N}^d} c_\alpha^2. \quad (6)$$

We remark that ρ is invariant under the orthogonal group action on \mathbf{R}^d , and moreover, the condition $\rho = 0$ is invariant even under all linear transformations of \mathbf{R}^d . Also it is easy to remark from Corollary 2.2 that the density of polynomials in $L^2((1+|x|^2)^2 d\mu(x))$ implies $\rho = 0$ (compare with Fuglede's untradedeterminacy condition [7]).

Another possible way of checking the uniqueness condition $\rho(\mathbf{a}) = 0$ is through the restriction of the moment sequences to the coordinate axes, as in [9]. To be more specific, let \mathbf{a} be the moment sequence of a positive measure μ on \mathbf{R}^d , and let \mathbf{a}_j , $1 \leq j \leq d$, be the induced boundary moment sequences:

$$\mathbf{a}_j(\alpha) = \mathbf{a}(0, \dots, 0, \alpha_j, 0, \dots, 0) = \int_{\mathbf{R}^d} x_j^{\alpha_j} d\mu(x), \quad \alpha \in \mathbf{N}^d.$$

Then, according to Theorem 3 of [9], if $\rho(\mathbf{a}_j) = 0$, $1 \leq j \leq d$, then $\rho(\mathbf{a}) = 0$. Moreover, the converse is also true in the case of product measures [9] Theorem 4. However, in general the converse is not valid, as shown by an example also contained in [9].

As expected, the condition $\rho = 0$ is not necessary, in general, for the unique determination of the representing measure. We present below such an example, adapted after Schmüdgen [12].

Proposition 2.3. *There exists a determinate moment sequence in two variables with $\rho \neq 0$.*

Proof. We closely follow the first example in [12]. Let μ be a positive measure on the real line which admits all moments and is indeterminate, yet N-extremal. That means the polynomials in one variable are dense in $L^2(\mu)$, but there exist other measures with the same moments, see also [11]. We define the measure $\nu = (1+x^2)^{-1}\mu$, so that ν is determinate (because for instance the multiplication by $(x+i)$ on polynomials has dense range in $L^2(\nu)$).

Let $j(x) = (\sqrt{2}x, x^2)$, $x \in \mathbf{R}$, be a fixed embedding of the line into \mathbf{R}^2 , and let $\sigma = j_*\nu$ be the image measure, supported by the parabola $2y = x^2$. Then it is easy to see that σ is a determinate measure, see [12].

Assume that the invariant ρ vanishes for the measure σ . This means that there exists a sequence of polynomials $p_n \in \mathbf{C}[x, y]$ satisfying:

$$\|(1+x^2+y^2)p_n - 1\|_{2,\sigma} \longrightarrow 0.$$

This in turn implies:

$$\|(1+x^2)^2 q_n - 1\|_{2,\nu} \longrightarrow 0,$$

where $q_n(x) = p(\sqrt{2}x, x^2)$. The last condition is equivalent to:

$$\|(x+i)(1+x^2)q_n(x) - \frac{1}{x-i}\|_{2,\mu} \longrightarrow 0. \quad (7)$$

Let V denote the closure of $(x+i)\mathbf{C}[x]$ in $L^2(\mu)$. Relation (7) shows that $\frac{1}{x-i} \in V$. Since the measure μ is indeterminate, for every $\epsilon > 0$ there exists a positive constant C with the property that:

$$|p(z)| \leq C e^{\epsilon|z|} \|p\|_{2,\mu}, \quad z \in \mathbf{C}, \quad p \in \mathbf{C}[x].$$

That is the evaluation at a given point $z \in \mathbf{C}$ is a bounded linear functional on the closure of polynomials, hence on all $L^2(\mu)$. Moreover, one can identify in this way $L^2(\mu)$ with a Hilbert space of entire functions of exponential type, see [11] or [1]. But this contradicts relation (7), because $\frac{1}{z-i}|_{z=-i} \neq 0$, while all elements of the space V vanish at the point $z = -i$.

□

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