PRIVILEGE ON STRICTLY CONVEX DOMAINS

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ABSTRACT. We adapt the privilege theorem of Douady and Pourcin from polydomains to strictly convex domains in the complex space.

1. INTRODUCTION

Let Ω be a bounded domain of $\mathbb{C}^n$, $n \geq 1$, and let $\mathcal{O}(\Omega)$ be the space of complex analytic functions in Ω. For a Banach algebra of analytic functions $B = B(\Omega) \subset \mathcal{O}(\Omega)$, we denote by $M(m,n;B)$ the space of $m \times n$ matrices with entries in B. Fix a matrix $d \in M(m,n;B)$ and $f \in B^n$. Division problems of the type:

$$du = f, \quad u \in B^m, \tag{1}$$

where u is the unknown, are fundamental in complex analysis. In this note we study conditions under which problem (1) is solvable if and only if the simpler equation, without control on the boundary behaviour, is solvable:

$$du' = f, \quad u' \in \mathcal{O}(\Omega)^m. \tag{2}$$

Early in the development of modern complex analytic geometry the importance of such division problems was singled out by A. Douady [3], [4], who applied them as a technical tool to deformation theory. Later on, developments and ramifications of his ideas have been considered by Pourcin [9] and Maltsiniotis [8]. It is interesting to remark that very recently, problems in electrical engineering led to similar division questions, see [2].

The aim of the present note is to study the division problem (1), and its restriction to (2), on strictly comvex domains of $\mathbb{C}^n$. This is done on the disk algebra of the domain, by using in an essential way the existance of peak functions at point in the boundary of the domain. The classical results of Douady and collaborators, proved on convex polydomains, appear in the globalization of the local data. The main theorem below asserts that, in the case B is the disk algebra, equation (1) is equivalent to (2), if and only if rk $d(z)$ is constant for $z \in \partial\Omega$. In its turn, this geometric condition has deep implications for the homology of the modules Coker d, and hence the possibility of classifying them by a manageable moduli space.

The first author was partially supported by the National Science Foundation Grant DMS 0100367; the second author supported by STINT, The Swedish Foundation for International Cooperation in Research and Higher Education.

2. Main result

Let Ω be a bounded, strictly convex domain of $\mathbb{C}^n$ and let $A(\Omega) = C(\overline{\Omega}) \cap O(\Omega)$, be the associated algebra of analytic functions in Ω which are continuous up to the boundary. Then $A(\Omega)$ is a commutative Banach algebra with 1 whose maximal ideal space coincides with $\overline{\Omega}$. We do not assume that the boundary of Ω is smooth.

A Banach module Y over $A(\Omega)$ is called *privileged* if there exists a resolution

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to A(\Omega)^{n_1} \xrightarrow{d_1} A(\Omega)^{n_0} \to Y \to 0,$$

where $d_q \in M(n_{q+1}, n_q;\ A(\Omega))$.

Lemma 2.1. *Suppose that Y is a privileged $A(\Omega)$-module with a resolution*

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to A(\Omega)^{n_1} \xrightarrow{d_1} A(\Omega)^{n_0} \to Y \to 0,$$

and that $a \in \partial\Omega$. Then the complex

$$0 \to \mathbb{C}^{n_p} \xrightarrow{d_p(a)} \cdots \to \mathbb{C}^{n_1} \xrightarrow{d_1(a)} \mathbb{C}^{n_0} \tag{3}$$

is exact.

Proof. Suppose that $\xi \in \text{Ker } d_k(a)$. Let $\varphi(z) \in A(\Omega)$ be a peak function for a, that is

$$|\varphi(z)| < 1, \quad z \in \overline{\Omega} \setminus \{a\}, \quad \varphi(a) = 1.$$

Since the map d_k has closed range, and

$$\lim_{n\to\infty} \|d_k(\varphi^n \xi)\|_{\infty,\Omega} = 0,$$

it follows that $\text{dist}(\varphi^n\xi, \text{Ker } d_k)$ converges to 0. Since $\text{Ker } d_k = \text{Im } d_{k+1}$, we can choose $\eta_n \in A(\Omega)^{n_{k+1}}$ such that

$$\|\varphi^n \xi - d_{k+1}\eta_n\|_\infty \to 0,$$

as $n \to \infty$. In particular

$$\xi = \lim_{n\to\infty} d_{k+1}(a)\eta_n(a),$$

and since $\text{Im } d_{k+1}(a) \subset \mathbb{C}^{n_k}$ is closed,

$$\xi \in \text{Im } d_{k+1}(a).$$

□

Thus, in the conditions of the lemma, for an arbitrary point $a \in \partial\Omega$, an Euler characteristic argument shows that:

$$\text{rk } d_1(a) = n_0 - n_1 + n_2 - \ldots + (-1)^p n_p.$$

Therefore, for a privileged module $Y = \text{Coker}(d_1)$, the function $\text{rk } d_1(a)$ is independent of a. The remarkable fact is that this rank condition is equivalent to privilege, and to other rather restrictive properties of the module.

The main result is stated as follows.

Theorem 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly convex domain and let $d \in M(n_1, n_0; A(\Omega))$. Then the following conditions are equivalent.*

(i) The range of d is closed.

(ii) The function $z \mapsto \mathrm{rk}\, d(z)$ is constant on $\partial\Omega$.

(iii) The natural morphism

$$\mathrm{Coker}\,(d : A(\Omega)^{n_1} \longrightarrow A(\Omega)^{n_0}) \mapsto \mathrm{Coker}\,(d : O(\Omega)^{n_1} \longrightarrow O(\Omega)^{n_0})$$

is injective.

(iv) There exists a finite, free resolution of finite type $A(\Omega)$-modules:

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} A(\Omega)^{n_1} \xrightarrow{d} A(\Omega)^{n_0} \to Coker\ d \to 0.$$

(v) There exists a linear continuously $\mathbb{C} -$ split resolution:

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} A(\Omega)^{n_1} \xrightarrow{d} A(\Omega)^{n_0} \to Coker\ d \to 0.$$

Proof. Trivially $(v) \Rightarrow (iv) \Rightarrow (i)$.

An adaptation of the proof of Lemma 2.1 yields $(i) \Rightarrow (ii)$. Indeed, assume that the map $d : A(\Omega)^{n_1} \longrightarrow A(\Omega)^{n_0}$ has closed range, and consider the exact sequence:

$$\mathrm{Ker}\ d \xrightarrow{i} A(\Omega)^{n_1} \xrightarrow{d} A(\Omega)^{n_0},$$

where i is the inclusion map. Fix a point $a \in \partial\Omega$ and a vector $\xi \in \mathrm{Ker}\ d(a)$. Arguing as in the proof of Lemma 2.1, there exists a sequence $\eta_n \in \mathrm{Ker}\ d$ with the property that $\lim_{n\to\infty}(\eta_n - \phi^n\xi) = 0$. In particular, $\lim_{n\to\infty}(\eta_n(a) - \xi) = 0$ in $\mathbb{C}^{n_1}$. Thus $\xi \in i(a)\mathrm{Ker}\ d$, since the inclusion and evaluation map $i(a) : \mathrm{Ker}\ d \longrightarrow \mathbb{C}^{n_1}$ has closed range. Remark that $i(a)$ depends continuosuly on a, in the operator norm.

Hence, for every point $a \in \partial\Omega$ we have proved the equality:

$$\mathrm{Im}\ i(a) = \mathrm{Ker}\ d(a).$$

But the dimensions of the two sides are lower, respectively upper semicontinuous. In conclusion the function $\dim \mathrm{Ker}\ d(a)$ is constant, therefore the rank of the matrix $d(a)$ is also constant for all $a \in \partial\Omega$.

By Cartan's theory, the module $\mathrm{Coker}(d : \mathcal{O}(\Omega)^{n_1} \longrightarrow \mathcal{O}(\Omega)^{n_0})$ carries a natural Fréchet space topology which makes the map (iii) continuous. Consequently, $(iii) \Rightarrow (i)$.

Proof of $(ii) \Rightarrow (v)$. Let E be the subset of $\overline{\Omega}$ where $d(z)$ has less than maximal rank m. The set E is analytic and contained in Ω, hence it is finite.

For every point $z \in \overline{\Omega} \setminus E$ there is a fundamental system of neighbourhoods U_z of z such that

$$\text{Coker}\ (d : A(U_z)^{n_1} \longrightarrow A(U_z)^{n_0}) \simeq A(U_z)^m. \tag{4}$$

Let $a \in E$ be a jumping point for the rank function. Then there exist polydisks U_a centered at a and relatively compact in Ω, so that

$$\text{rk}\ d(z) = m, \quad z \in \overline{U_a} \setminus \{a\}.$$

By Douady's privilege theorem, [3] Proposition 6, or [9] Theorem 3.2, there exists a topologically $\mathbb{C}$-split resolution:

$$0 \to A(U_a)^{m_q} \xrightarrow{d_q} \cdots \xrightarrow{d_2} A(U_a)^{m_1} \xrightarrow{d} A(U_a)^{m_0} \to \text{Coker}\ d|_{A(U_a)} \to 0, \tag{5}$$

where the numbers $m_0, m_1, \dots, m_q$ may depend on a.

Next we divide the space by a rectangular grid, parallel to the real coordinate axes, so that each cell intersects $\overline{\Omega}$ into a convex set contained in one of the neighbourhoods $U_z,\ \ z \in \overline{\Omega} \setminus E$, or $U_a,\ \ a \in E$. Since the domain Ω was supposed to be convex, a repeated application of Cartan's lemma of invertible matrices (see [3] Théorème 6.3) allows us to "glue" together the resolutions (4) or (5) into a single, topologically $\mathbb{C}$-split, finite free reolution of $\text{Coker}\, d$, as in the statement (5).

Proof of $(v) \Rightarrow (iii)$. This part of the proof requires a localization argument. We repeat, with minor modifications, the corresponding proof in [9].

Let $U \subset \mathbb{C}^n$ be an arbitrary Stein open set, and let:

$$\mathcal{A}(U) = \mathcal{O}(U \cap \Omega) \cap C(U \cap \overline{\Omega}).$$

The existence of uniform estimates for the $\overline{\partial}$-equation ([7]) prove that $\mathcal{A}$ is a sheaf of Fréchet modules over $\mathcal{O}$, with

$$H^0(\mathbb{C}^n, \mathcal{A}) = A(\Omega), \quad \text{supp}\, \mathcal{A} = \overline{\Omega},$$

and

$$\mathcal{A}|_{\Omega} = \mathcal{O}_{\Omega}.$$

Moreover, $\mathcal{A}$ is acyclic on Stein open subsets of $\mathbb{C}^n$. For a proof of all these statements see [5] Chapter 4.

We assume that the complex:

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} A(\Omega)^{n_1} \xrightarrow{d} A(\Omega)^{n_0} \to Y \to 0 \tag{6}$$

is split (topologically and $\mathbb{C}$-linearly), where $Y = \text{Coker}\ d$. We can assume that $0 \in \Omega$, so that, by convexity, the homothethy $z \mapsto rz,\ 0 < r \leq 1$, maps Ω into itself.

Due to the stability of exactness of split complexes under small perurbations of the boundaries, see for instance [5], there exists $\epsilon > 0$ with the property that the complex:

$$0 \to A(\Omega)^{n_p} \xrightarrow{d_p(rz)} \cdots \xrightarrow{d_2(rz)} A(\Omega)^{n_1} \xrightarrow{d(rz)} A(\Omega)^{n_0}$$

is still exact and has separated 0-th order homology, for every $z \in \Omega$ and $1-\epsilon < r < 1$. But this is equivalent to the fact that the complex

$$0 \to A(r\Omega)^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} A(r\Omega)^{n_1} \xrightarrow{d} A(r\Omega)^{n_0}$$

is exact and has separated 0-th order homology for $1-\epsilon < r < 1$.

Since the topology of the nuclear Fréchet space $\mathcal{O}(\Omega)$ can be given by the sup-norms on $r\overline{\Omega}$, $r < 1$, a Mittag-Leffler type argument (and Cartan's theorem A) shows that the complex of sheaves:

$$0 \to {\mathcal{O}_\Omega}^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} {\mathcal{O}_\Omega}^{n_1} \xrightarrow{d} {\mathcal{O}_\Omega}^{n_0}$$

is exact.

On the other hand, since rk $d(z)$ is constant for z close to the boundary of Ω, for every point $a \in \partial\Omega$, there exists a fundamental system of Stein neighbourhoods U_a, with the property that the complex:

$$0 \to A(U_a)^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} A(U_a)^{n_1} \xrightarrow{d} A(U_a)^{n_0}$$

is split exact.

Thus, by putting together this local analysis, we infer that the complex of Fréchet analytic sheaves

$$0 \to \mathcal{A}^{n_p} \xrightarrow{d_p} \cdots \xrightarrow{d_2} \mathcal{A}^{n_1} \xrightarrow{d} \mathcal{A}^{n_0} \to \mathcal{Y} \to 0$$

is exact.

An application of the long exact sequence of cohomology shows that $\mathcal{Y} = \text{Coker}\,(d : \mathcal{A}^{n_1} \longrightarrow \mathcal{A}^{n_0})$ is acylic on Stein open sets, it is supported by $\overline{\Omega}$ and

$$H^0(\mathbb{C}^n, \mathcal{Y}) = Y.$$

Moreover, $\mathcal{Y}$ is locally $\mathcal{A}$-free on an open neighbourhood of $\partial\Omega$. In conclusion, the restriction map

$$\rho : \mathcal{Y} \longrightarrow \mathcal{Y}|_\Omega,$$

is injective. And so it remains at the level of global sections:

$$\rho : H^0(\mathbb{C}^n, \mathcal{Y}) \longrightarrow H^0(\Omega, \mathcal{Y}).$$

Finally, since:

$$\text{Coker}\,(d : A(\Omega)^{n_1} \longrightarrow A(\Omega)^{n_0}) = H^0(\mathbb{C}^n, \mathcal{Y})$$

and

$$\text{Coker}\,(d : \mathcal{O}(\Omega)^{n_1} \longrightarrow O(\Omega)^{n_0}) = H^0(\Omega, \mathcal{Y}),$$

assertion (iii) follows.

This finishes the proof of the theorem. □

To illustrate the theorem, we give a simple application, in the spirit of the Gleason problem.

Proposition 2.3. *Let Ω be a bounded strictly convex domain in $\mathbb{C}^n$ and let $f_1, f_2, \ldots, f_k \in A(\Omega)$, not all identically equal to zero. The map:*

$$(f_1, f_2, ..., f_k) : A(\Omega)^k \longrightarrow A(\Omega)$$

has closed range if and only if:

$$\inf_{a \in \partial\Omega} (|f_1(a)| + \ldots + |f_k(a)|) > 0. \tag{7}$$

In this case a function $g \in A(\Omega)$ belongs to the ideal generated by $f_1, f_2, \ldots, f_k$ in $A(\Omega)$ if and only if:

$$g_z \in (f_1, f_2, \ldots, f_k)\mathcal{O}_z,$$

in the sense of germs of analytic functions, for every common zero $z \in \Omega$: $f_1(z) = f_2(z) = \ldots = f_k(z) = 0$.

Similarly, for every k-tuple of non-identically to zero functions $f_1, f_2, \ldots, f_k$ in $A(\Omega)$, the map:

$$(f_1, f_2, ..., f_k)^T : A(\Omega) \longrightarrow A(\Omega)^k,$$

has closed range if and only if the same non-vanishing condition (7) holds.

It would be interesting to know whether conditions $(i)-(v)$ in the theorem are also equivalent to the closure of the range of $d : X(\Omega) \longrightarrow X(\Omega)$, where $X(\Omega)$ is the Bergman, Hardy, or a similar Banach space of analytic functions on Ω.

To end, we note that the main result of this note does not hold on non-strictly convex domains, as for instance the case of a polydisk shows, see [9]. Also, the corresponding picture for the algebra $A^\infty(\Omega) = \mathcal{C}^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ is quite different, cf. [1].

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