On bounded analytic extension in $\mathbf{C}^n$ *†

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Paper dedicated to Ion Colojoară on the occasion of his seventieth birthday

Introduction. Let Ω be a domain in $\mathbf{C}^n$, $n \geq 1$, and let V be a complex analytic subset of Ω. The question whether a function defined along V can be extended analytically to the whole Ω was well understood by cohomological methods half a century ago. The corresponding problem for classes of analytic functions satisfying growth conditions is recognized to be more intricate. Partial results in this direction, for instance for bounded analytic functions, were obtained by integral represenation methods and they are true only under additional smoothness or transversality conditions, see Amar[Am1], Henkin and Leiterer [HL], Polyakov [Po], Rudin [Ru1, Ru2].

Of a more recent date is the multivariable extension of the Hilbert space methods used in the Nevanlinna parametrization of the solution to the power moment problem in one variable. Following the pioneering work of Jim Agler [Ag],[AC] several authors have proved and applied in this new spirit interpolation theorems (à la Nevanlinna-Pick) for special domains of $\mathbf{C}^n$, see [AM], [BT], [CLW], [He]. An important conclusion of these works is that for bounded interpolation theory in several complex variables the class $H^\infty(\Omega)$ of all bounded analytic functions in Ω is rather big and sometimes inappropriate; the correct class turns out to be a subspace of $H^\infty(\Omega)$, called the Schur algebra $\mathcal{S}(\Omega)$ (to be defined below). Only in this restrictive way the essential parts of the classical dilation theory (including von-Neumann's inequality and the basic interpolation applications) were extended to the multidimensional setting, see[AP], [Ar], [DP].

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The aim of the present note is to provide new proofs, in the spirit of state space realization in linear systems theory, of some known rigidity phenomena of bounded analytic extension. Namely the fact that in the case V is an analytic disk attached to a ball or a polydisk Ω in $\mathbf{C}^n$, $n > 1$, the hypothesis that analytic extension from $H^\infty(V) = \mathcal{S}(V)$ to $\mathcal{S}(\Omega)$ exists without norm increase implies a very restrictive geometry of the pair $V \subset \Omega$. To be more specific, if $\Omega = B$ is a ball, then V must be the intersection of B with an affine line, while if $\Omega = D_1 \times D_2 \times \ldots D_n$ is a polydisk, then we show that V must be the graph of an analytic function defined on one of its factors D_j.

The original proofs of similar and more general results about analytic extension from H^∞ to H^∞ invoked the Schwarz Lemma and some specific geometric arguments, see Section 7.5 in [Ru1] and Section 8.3 in [Ru2]. We hope that our Hilbert space approach will find applications to related questions or it will offer natural generalizations to other domains.

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PRELIMINARIES. Let $\mathbf{B}$ be the unit ball in $\mathbf{C}^n$, $n \geq 1$. The Hermitian scalar product in $\mathbf{C}^n$ will be denoted by $\langle \cdot, \cdot \rangle$ and the unit disk in $\mathbf{C}$ by $\mathbf{D}$. By definition, an analytic function f belongs to the *Schur class* $\mathcal{S}(\mathbf{B})$ if there exists a positive constant R so that the kernel:

$$\frac{R^2 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}, \quad z, w \in \mathbf{B},$$

is non-negative definite. The Schur norm $\|f\|_{\mathcal{S}}$ of f is then the optimal constant staisfying this positivity requirement. In this case, for every system of commuting operators $T = (T_1, T_2, \ldots, T_n)$ subject to a spherical constraint: $T_1^*T_1 + T_2^*T_2 + \ldots T_n^*T_n \leq cI$, $c < 1$, von-Neumann's inequality holds: $\|f(T)\| \leq \|f\|_{\mathcal{S}}$. Note that, in general $\|f\|_{\infty,\mathbf{B}} < \|f\|_{\mathcal{S}}$, and that von-Neumann's inequality with respect to the uniform norm fails in the multidimansional case of three or more variables. For function theorists one can define the Schur class as the set of all bounded analytic multipliers of the Hilbert space with reproducing kernel $\frac{1}{1-\langle z, \overline{w} \rangle}$, $z, w \in \mathbf{B}$, see for details [Ar].

In this new framework the classical Nevanlinna-Pick interpolation theorem has a direct counterpart, with the expected positivity conditions imposed to the above kernel. To be more specific:

If A is a subset of $\mathbf{B}$ and a function $f : A \longrightarrow \mathbf{D}$ is given so that the kernel

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}, \quad z, w \in A,$$

is non-negative definite, then f extends to an analytic function $F : \mathbf{B} \longrightarrow \mathbf{D}$.

For more details see [AC], [Ar], [BT], [EP].

Let us finally remark that the restriction of the Schur class $\mathcal{S}(\mathbf{B})$ to the intersection of the unit ball $\mathbf{B}$ with a complex linear variety V is the Schur class $\mathcal{S}(\mathbf{B} \cap V)$, with respect to the restricted Hermitian metric on the ball $\mathbf{B} \cap V$.

In the case of the unit polydisk $\mathbf{D}^n \subset \mathbf{C}^n$, the Schur class $\mathcal{S}(\mathbf{D}^n)$ consists of those bounded analytic functions $f \in H^\infty(\mathbf{D}^n)$ satisfying:

$$\|f\|_{\mathcal{S}} = \sup_{T,r} \|f(rT)\| < \infty$$

where T is an arbitrary n-tuple of commuting contractions and $r < 1$. As in the case of the ball, the Schur norm can be defined by a positivity condition (less constructive this time) as follows: there exist $R > 0$ and positive definite kernels K_j such that:

$$R^2 - f(z)\overline{f(w)} = \sum_{j=1}^{n} (1 - z_j \overline{w}_j) K_j(z, \overline{w}), \quad z, w \in \mathbf{D}^n.$$

The norm $\|f\|_{\mathcal{S}}$ is then equal to the optimal constant R, see [Ag].

If $n = 1$, von-Neumann's inequality implies $\|f\|_{\mathcal{S}} = \|f\|_\infty$ and hence $\mathcal{S}(\mathbf{D}) = H^\infty(\mathbf{D})$, isometrically. Due to Ando's Dilation Theorem the case $n = 2$ is also special: $\mathcal{S}(\mathbf{D}^2) = H^\infty(\mathbf{D}^2)$. For higher dimensions however, it is known that $\mathcal{S}(\mathbf{D}^n) \neq H^\infty(\mathbf{D}^n)$, $n > 2$. For details see [FF].

The following elementary lemma will be needed in the extension results.

Lemma 1. *Let $K_1, K_2, \ldots, K_d$ be non-negative definite kernels, defined on a set I.*

a) If

$$K_1(i,j) + K_2(i,j) + \ldots + K_d(i,j) = 1, \quad i, j \in I,$$

then each $K_l, 1 \leq l \leq d$, is constant: $K_l(i,j) = c_l,\ i, j \in I$.

b) Suppose that

$$K_1(i,j)K_2(i,j) = 1, \quad i,j \in I.$$

Then there is a function $f : I \longrightarrow \mathbf{C}$ satisfying:

$$K_2(i,j) = f(i)\overline{f(j)}, \quad i,j \in I.$$

Proof. a). Let $c_i \in \mathbf{C}$ be a finitely supported function of $i \in I$. According to our assumptions we have for each $l \leq d$:

$$\sum_{i,j} K_l(i,j)c_i\overline{c_j} \leq |\sum_i c_i|^2.$$

Let $K_l(i,j) = \langle F_i, F_j\rangle$ be the Kolmogorov factorization, where $F : I \longrightarrow H$ is a Hilbert space valued function. Thus,

$$\|\sum_i c_i F_i\| \leq |\sum_i c_i|,$$

for all finitely supported functions c_i. This shows that we can take $H = \mathbf{C}$ and all F_i equal to a non-negative constant.

b). If the set I has a single element, then we have nothing to prove. Let $J = \{a,b\}$ be a subset of I consisting of exactly two elements. Then the matrix

$$\begin{pmatrix} K_1(a,a) & K_1(a,b) \\ K_1(b,a) & K_1(b,b) \end{pmatrix}$$

has non-zero entries and it is non-negative definite. Therefore its determinant Δ is non-negative.

On the other hand the Schur inverse matrix:

$$\begin{pmatrix} K_1^{-1}(a,a) & K_1^{-1}(a,b) \\ K_1^{-1}(b,a) & K_1^{-1}(b,b) \end{pmatrix}$$

has the same properties. The determinant of this matrix is, up to a positive scalar, equal to $-\Delta$; and it is non-negative. Hence $\Delta = 0$.

Thus the rank of the kernel $K_1(i,j), \quad i,j \in I$, is equal to one. Which means that a factorization as in the statement exists.

Henceforth, by an *analytic disk* attached to a domain $\Omega \subset \mathbf{C}^n$ we mean an analytic, proper, injective map $\phi : \mathbf{D} \longrightarrow \Omega$. If Ω is bounded then the non-tangential limits of ϕ exist almost everywhere with respect to the linear measure on the unit circle $\partial\mathbf{D}$ and the assumption that ϕ is proper implies that these boundary values belong to the boundary of Ω.

EXTENSION TO THE BALL. The main result of this section is stated below.

Theorem 2. *Let* $\mathbf{B}$ *be the unit ball in* $\mathbf{C}^n, n > 1$, *and let* $V \subset \mathbf{B}$ *be an analytic disk attached to it. Assume that every analytic function* $f \in H^\infty(V)$ *can be extended to an analytic function* $F \in \mathcal{S}(\mathbf{B})$ *with* $\|F\|_{\mathcal{S}} = \|f\|_\infty$.

Then V *is the intersection of a complex line with* $\mathbf{B}$.

Proof. Let W be a complex affine manifold and denote $W_0 = W - W$. Then we claim that each function $f \in \mathcal{S}(W \cap \mathbf{B})$ can be analytically extended to $\mathcal{S}(\mathbf{B})$ without norm increase.

Indeed, let v_0 be the closest element to 0 in W, so that we can parametrize:

$$W = \{v_0 + u; \quad u \in W_0, \quad u \perp v_0\}.$$

Therefore

$$W \cap \mathbf{B} = \{v_0 + u; \quad u \in W_0, \ \|u\|^2 < 1 - \|v_0\|^2\}.$$

A function $f \in \mathcal{S}(W \cap \mathbf{B})$ of Schur norm equal to 1 has a non-negative corresponding kernel:

$$K(u, \overline{v}) = \frac{1 - f(v_0 + u)\overline{f(v_0 + v)}}{1 - \|v_0\|^2 - \langle u, v\rangle}.$$

But

$$K(u, \overline{v}) = \frac{1 - f(v_0 + u)\overline{f(v_0 + v)}}{1 - \langle v_0 + u, v_0 + v\rangle},$$

so that the above mentioned version of Nevanlinna-Pick theorem assures the existence of an analytic extension to $\mathcal{S}(\mathbf{B})$ of norm less or equal to 1.

To prove the converse direction, let $\phi : \mathbf{D} \longrightarrow V$ be a proper analytic isomorphism. Let $\psi : V \longrightarrow \mathbf{D}$ be the inverse map.

The statement is invariant under holomorphig automorphisms (i.e. Möbius transforms) of the ball. Indeed, such a variabble change ($z \mapsto \frac{z-a}{1-\langle z,a\rangle}$, with $a \in \mathbf{B}$) maps complex linear varieties into complex linear varieties (see also

Proposition 2.4.2 in [Ru2]) and it leaves invariant the Schur class, too. Thus we can assume that $\phi(0) = 0$.

Since $\psi \in H^\infty(V)$ nas norm one, it has by hypothesis an extension of norm one in $\mathcal{S}(\mathbf{B})$. Thus the kernel:

$$\frac{1 - \psi(\phi(u))\overline{\psi(\phi(v))}}{1 - \langle \phi(u), \phi(v) \rangle} = \frac{1 - u\overline{v}}{1 - \langle \phi(u), \phi(v) \rangle}, \quad u, v \in \mathbf{D},$$

is non-negative definite.

On the other hand, the map $\phi : \mathbf{D} \longrightarrow \mathbf{B}$ is analytic, hence by the one-variable (vector valued) Nevanlinna-Pick criterion, the kernel:

$$\frac{1 - \langle \phi(u), \phi(v) \rangle}{1 - u\overline{v}}$$

is non-negative definite, too.

According to Lemma 1, there exists a function $f : \mathbf{D} \longrightarrow \mathbf{C}$ such that:

$$\frac{1 - u\overline{v}}{1 - \langle \phi(u), \phi(v) \rangle} = f(u)\overline{f(v)}, \quad u, v \in \mathbf{D}.$$

By taking $u = 0$ in the latter identity we find that $f(v)$ is constant and actually $f(u)\overline{f(v)} = 1$ for all $u, v \in \mathbf{D}$.

Thus,

$$1 - u\overline{v} = 1 - \langle \phi(u), \phi(v) \rangle, \quad u, v \in \mathbf{D}.$$

By developing ϕ into its Taylor series at the origin:

$$\phi(u) = \sum_{j=0}^{\infty} c_j u^j, \quad c_j \in \mathbf{C}^n,$$

we obtain the power series identity:

$$u\overline{v} = \sum_{i,j}^{\infty} \langle c_i, c_j \rangle u^i \overline{v}^j,$$

which implies $c_j = 0$ for all $j \neq 1$.

In conclusion ϕ must be a linear map of the form $\phi(u) = cu$ for some unit vector $c \in \mathbf{C}^n$.

EXTENSION TO THE POLYDISK. The proof of the corresponding result in the case of a polydisk is entirely similar. First, the statement.

Theorem 2. *Let* $\mathbf{D}^n, n > 1$, *be the unit polydisk and let* V *be an analytic disk attached to it. If every analytic function* $f \in H^\infty(V)$ *extends to* $\mathcal{S}(\mathbf{D}^n)$ *without norm increase, then* V *must be the graph of an analytic function defined on one of its factors.*

Proof. In one direction, it is obvious that, assuming $V = \{(u, H(u)); u \in \mathbf{D}\}$, with a given analytic map $H : \mathbf{D} \longrightarrow \mathbf{D}^{n-1}$, one can extend any bounded function $f(u, H(u))$ along V to the whole polydisk by the formula $F(u, v) = f(u, H(u)), \quad u \in \mathbf{D}, v \in \mathbf{D}^{n-1}$.

Conversely, assume that $\phi = (\phi_1, \phi_2, \ldots, \phi_n) : \mathbf{D} \longrightarrow V$ is a parametrization of the analytic disk. Let $\psi : V \longrightarrow \mathbf{D}$ be its inverse (analytic) map.

Since the statement is invariant under analytic automorphisms of the polydisk, we can assume that $\phi(0) = 0$.

By our hypothesis, ψ extends analytically to the Schur class $\mathcal{S}(\mathbf{D}^n)$ and the extension has norm one there. Thus, by Nevanlinna-Pick theorem, there exist non-negative definite kernels $K_j, 1 \leq j \leq n$, so that:

$$1 - u\overline{v} = 1 - \psi(\phi(u))\overline{\psi(\phi(v))} = \sum_{j=1}^{n} (1 - \phi_j(u)\overline{\phi_j(v)}) K_j(u, \overline{v}), \quad u, v \in \mathbf{D}.$$

Some of the terms in this decomposition might be identically zero. Let us suppose that K_1 is not identically equal to zero.

Since each map $\phi_j : \mathbf{D} \longrightarrow \mathbf{D}, 1 \leq j \leq n$, is analytic, the classical Nevanlinna-Pick theorem implies that each of the kernels:

$$\frac{1 - \phi_j(u)\overline{\phi_j(v)}}{1 - u\overline{v}},$$

is non-negative definite.

By Lemma 1 and the above identity we find that there exists a function $f : \mathbf{D} \longrightarrow \mathbf{C}$ such that

$$\frac{1 - \phi_1(u)\overline{\phi_1(v)}}{1 - u\overline{v}} = f(u)\overline{f(v)}, \quad u, v \in \mathbf{D}.$$

At this point we can repeat the argument in the proof of Theorem 2 and infer that ϕ_1 is the linear map: $\phi_1(u) = e^{i\theta}u, \ \theta \in \mathbf{R}$.

Thus, in view of Ando's Theorem, for $n = 2$ one can replace $\mathcal{S}(\mathbf{D}^2)$ by $H^\infty(\mathbf{D}^2)$ in the statement of Theorem 2.

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