Math 240B: Problem Set V

February 27, 2009

Due: Friday, March 6, 2009.

Exercise VIII. a. Suppose that X is a smooth vector field on M. Define the *interior product* $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$\iota_X(\omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}).$$

Show that if $\omega \in \Omega^k(M)$ and $\theta \in \Omega^l(M)$, then

$$\iota_X(\omega \wedge \theta) = (\iota_X \omega) \wedge \theta + (-1)^k \omega \wedge (\iota_X \theta).$$

b. Define a real linear operator $L_X : \Omega^k(M) \to \Omega^k(M)$ by

$$L_X = d \circ \iota_X + \iota_X \circ d.$$

Show that if $\omega \in \Omega^k(M)$ and $\theta \in \Omega^l(M)$, then

$$L_X(\omega \wedge \theta) = (L_X \omega) \wedge \theta + \omega \wedge (L_X \theta).$$

We call L_X the *Lie derivative* in the direction of X.

Exercise IX. Use the Mayer-Vietoris sequence to determine the de Rham cohomology of the two-sphere Σ_g with g handles, the compact oriented surface of genus g. Hint: Use the fact that Σ_g is orientable and therefore has a volume form which makes the top cohomology nonzero.