

Math 240B: Problem Set IV

February 13, 2009

Due: Monday, February 23, 2009.

Exercise VI. Show that if X and Y are smooth vector fields on M and $\theta \in \Omega^1(M)$ is a smooth one-form, then

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]).$$

Hint: Use local coordinates.

Exercise VII. For $1 \leq i, j \leq n$, define functions $x_j^i, y_j^i : GL(n, \mathbb{R}) \rightarrow \mathbb{R}$ by

$$x_j^i \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \cdot & \cdots & \cdot \\ a_1^n & \cdots & a_n^n \end{pmatrix} = a_j^i, \quad y_j^i = x_j^i(A^{-1}).$$

a. Show that the differential form

$$\omega_j^i = \sum_{k=1}^n y_k^i dx_j^k$$

is left invariant.

b. Establish the *Maurer-Cartan equations* for $GL(n, \mathbb{R})$:

$$d\omega_j^i = - \sum_{k=1}^n \omega_k^i \wedge \omega_j^k.$$