Math 240B: Problem Set III
Solution

February 18, 2009

Exercise V. We consider the special case of the construction in §1.13 of the lecture notes in which $G = U(n)$ and s is conjugation with

$$I_{1,n-1} = \begin{pmatrix} -1 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix},$$

so that the fixed point set of the automorphism s is $H = U(1) \times U(n-1)$ and $G/H = \mathbb{C}P^{n-1}$.

a. Recall that the Lie algebra $\mathfrak{u}(n)$ divides into a direct sum $\mathfrak{u}(n) = \mathfrak{h} \oplus \mathfrak{p}$, where

$$\mathfrak{h} = \{X \in \mathfrak{g} : s_*(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : s_*(X) = -X\},$$

where $\mathfrak{h}$ is the Lie algebra of $U(1) \times U(n-1)$. Consider two elements

$$X = \begin{pmatrix} 0 & -\bar{\xi}_2 & \cdots & -\bar{\xi}_n \\ \xi_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \xi_n & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -\bar{\eta}_2 & \cdots & -\bar{\eta}_n \\ \eta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \eta_n & 0 & \cdots & 0 \end{pmatrix}$$

of $\mathfrak{p}$, and determine their Lie bracket $[X, Y] \in \mathfrak{h}$.

Solution: It is easier if you carry out the multiplication in matrix terms. To simplify notation, write

$$X = \begin{pmatrix} 0 & -\bar{\xi}^T \\ \xi & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -\bar{\eta}^T \\ \eta & 0 \end{pmatrix}, \tag{1}$$

so that

$$XY = \begin{pmatrix} -\bar{\xi}^T\eta & 0 \\ 0 & -\xi\bar{\eta}^T \end{pmatrix}, \quad YX = \begin{pmatrix} -\bar{\eta}^T\xi & 0 \\ 0 & -\eta\bar{\xi}^T \end{pmatrix}$$

and

$$[X, Y] = \begin{pmatrix} -\bar{\xi}^T\eta + \bar{\eta}^T\xi & 0 \\ 0 & -\xi\bar{\eta}^T + \eta\bar{\xi}^T \end{pmatrix}.$$

b. Use the formula for curvature of G/H to show that the sectional curvatures $K(\sigma)$ for $\mathbb{C}P^{n-1}$ satisfy the inequalities $a^2 \leq K(\sigma) \leq 4a^2$ for some $a^2 > 0$.

Solution: As inner product on $T_I U(n)$, we use

$$\langle A, B \rangle = \frac{1}{2} \mathrm{Re} \left(\mathrm{Trace}(A^T \bar{B}) \right), \quad \text{for} \quad A, B \in \mathfrak{u}(n).$$

This differs by a factor of four from the Riemannian metric induced by the natural imbedding into $\mathbb{E}^{(2n)^2}$, but with the rescaled metric

$$\langle X, Y \rangle = \mathrm{Re}(\xi^T \bar{\eta}),$$

when X and Y are given by (1). To simplify the calculations, assume that

$$\langle X, X \rangle = \langle Y, Y \rangle = 1, \quad \text{and} \quad \langle X, Y \rangle = 0.$$

Then

$$|\xi|^2 = |\eta|^2 = 1 \quad \text{and} \quad \xi^T \bar{\eta} = -\eta^T \bar{\xi},$$

the latter since $\xi^T \bar{\eta}$ is purely imaginary. Then

$$\begin{aligned}
&\langle [X, Y], [X, Y] \rangle \\
&\quad = \frac{1}{2} \mathrm{Trace} \begin{pmatrix} -\eta^T \bar{\xi} + \xi^T \bar{\eta} & 0 \\ 0 & -\bar{\eta} \xi^T + \bar{\xi} \eta^T \end{pmatrix} \begin{pmatrix} -\xi^T \bar{\eta} + \eta^T \bar{\xi} & 0 \\ 0 & -\bar{\xi} \eta^T + \bar{\eta} \xi^T \end{pmatrix} \\
&\qquad = \frac{1}{2} \mathrm{Trace} \begin{pmatrix} 4 \left| \mathrm{Im}(\xi^T \bar{\eta}) \right|^2 & 0 \\ 0 & (-\bar{\eta} \xi^T + \bar{\xi} \eta^T)(-\bar{\xi} \eta^T + \bar{\eta} \xi^T) \end{pmatrix} \\
&\qquad\qquad = 2 \left| \mathrm{Im}(\xi^T \bar{\eta}) \right|^2 + |\xi|^| \eta|^2 + \left| \mathrm{Im}(\xi^T \bar{\eta}) \right|^2 \\
&\qquad\qquad\qquad = |\xi|^| \eta|^2 + 3 \left| \mathrm{Im}(\xi^T \bar{\eta}) \right|^2 .
\end{aligned}$$

The last expression ranges between 1 and 4, and it follows from the Cauchy-Schwarz inequality that it achieves its maximum when $\eta = i\xi$. Thus if σ is the two-plane spanned by X and Y,

$$K(\sigma) = \frac{4 \langle [X, Y], [X, Y] \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2} = 4 \left[|\xi|^| \eta|^2 + 3 \left| \mathrm{Im}(\xi^T \bar{\eta}) \right|^2 \right]$$

lies in the interval $[4, 16]$, achieving both extreme values when $n - 1 \geq 2$.