

# Math 240B: Problem Set III

February 6, 2009

Due: Friday, February 13, 2009.

**Exercise V.** We consider the special case of the construction in §1.13 of the lecture notes in which  $G = U(n)$  and  $s$  is conjugation with

$$I_{1,n-1} = \begin{pmatrix} -1 & 0 \\ 0 & I_{(n-1) \times (n-1)} \end{pmatrix},$$

so that the fixed point set of the automorphism  $s$  is  $H = U(1) \times U(n-1)$  and  $G/H = \mathbb{C}P^{n-1}$ .

a. Recall that the Lie algebra  $\mathfrak{u}(n)$  divides into a direct sum  $\mathfrak{u}(n) = \mathfrak{h} \oplus \mathfrak{p}$ , where

$$\mathfrak{h} = \{X \in \mathfrak{g} : s_*(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : s_*(X) = -X\},$$

where  $\mathfrak{h}$  is the Lie algebra of  $U(1) \times U(n-1)$ . Consider two elements

$$X = \begin{pmatrix} 0 & -\bar{\xi}_2 & \cdots & -\bar{\xi}_n \\ \xi_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \xi_n & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -\bar{\eta}_2 & \cdots & -\bar{\eta}_n \\ \eta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \eta_n & 0 & \cdots & 0 \end{pmatrix}$$

of  $\mathfrak{p}$ , and determine their Lie bracket  $[X, Y] \in \mathfrak{h}$ .

b. Use the formula for curvature of  $G/H$  to show that the sectional curvatures  $K(\sigma)$  for  $\mathbb{C}P^{n-1}$  satisfy the inequalities  $a^2 \leq K(\sigma) \leq 4a^2$  for some  $a^2 > 0$ .

**Remark.** The Riemannian metric defined on  $G/H = \mathbb{C}P^{n-1}$  is called the *Fubini-Study metric*. It occurs frequently in algebraic geometry.