Math 240B: Problem Set I

January 21, 2009

Due: Wednesday, January 28, 2009.

Exercise I. Suppose that M^2 is the right circular cylinder defined by the equation $x^2 + y^2 = 1$ in $\mathbb{E}^3$. Show that for each choice of real numbers a and b the curve

$$\gamma : \mathbb{R} \to M^2 \subseteq \mathbb{E}^3 \quad \text{defined by} \quad \gamma_{a,b}(t) = \begin{pmatrix} \cos(at) \\ \sin(at) \\ bt \end{pmatrix}$$

is a geodesic.

Exercise II. Consider the upper half-plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy),$$

the so-called Poincaré upper half plane.

a. Calculate the Christoffel symbols Γ^k_{ij}.

b. Write down the equations for the geodesics, obtaining two equations

$$\frac{d^2x}{dt^2} = \cdots , \quad \frac{d^2y}{dt^2} = \cdots .$$

c. Assume that $y = y(x)$ and eliminate t from these two equations by using the relation

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}.$$

Solve the resulting differential equation to determine the paths traced by the geodesics in the Poincaré upper half plane.