**Mathematics 5B Spring 2011: Review for Final**
June 3, 2011
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**Part I. Multiple Choice.** There will be several multiple choice questions in which you need to circle the correct answer. These will be based upon the lectures as well as questions from WebWork. Examples of multiple choice questions can be found in the practice quizzes which are posted online at:
http://math.ucsb.edu/ moore/s5bs2011.html
(Drafts of the lectures are also posted here.)

**Part II.** There will also be a section in which you need to give complete answers to questions. There will be several questions similar to the ones on the midterms. Here are some typical questions on the most recent material:

1. a. Find the divergence of the vector field

$$\mathbf{F}(x,y,z) = \log(y^2+z^2+1)\mathbf{i} + 3y\mathbf{j} + (\sin x \cos y)\mathbf{k}.$$

Answer: $\nabla \cdot \mathbf{F} = 3$.

b. Use the divergence theorem and the change of variables formula for a multiple integral to evaluate the flux integral

$$\int\int_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N} dA = \int\int_{\mathbf{S}} \mathbf{F}\cdot\mathbf{dS},$$

where $\mathbf{S}$ is the boundary of the region

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{(x-5)^2}{4} + \frac{(y+1)^2}{9} + \frac{(z-2)^2}{16} \leq 1 \right\}$$

and $\mathbf{N}$ is the outward-pointing unit normal.

Partial solution: From the divergence theorem, we conclude that

$$\int\int_{\mathbf{S}} \mathbf{F}\cdot\mathbf{N} dA = \int\int\int_D (\nabla\cdot\mathbf{F}) dxdydz = \int\int\int_D 3dxdydz.$$

We now make the change of variables

$$\frac{(x-5)}{2} = u, \qquad \frac{(y+1)}{3} = v, \qquad \frac{(z-2)}{4} = w.$$

We solve for $x$, $y$ and $z$ to obtain

$$x = 5+2u, \qquad y = -1+3v, \qquad z = 2+4w.$$

We then find that the determinant of Jacobian matrix is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 24.$$

Let

$$E = \left\{ (u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1 \right\}.$$

Then

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA = \int\int\int_E 3 \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

$$= \int\int\int_E 72 \, dudvdw = 96\pi.$$

2. a. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = e^{-y^2 - z^2}\mathbf{i} - 5yz^4\mathbf{j} + (z^5 - 1)\mathbf{k}.$$

Answer: $\nabla \cdot \mathbf{F} = 0$.

b. Evaluate the flux integral

$$\int\int_{\mathbf{S_1}} \mathbf{F} \cdot \mathbf{N} dA = \int\int_{\mathbf{S_1}} \mathbf{F} \cdot \mathbf{dS},$$

where $\mathbf{S_1}$ is the part of the $(x, y)$-plane bounded by the unit circle $x^2 + y^2 = 1$ and $\mathbf{N}$ is the downward-pointing unit normal.

Partial solution: First note that along the $(x, y)$-plane, $\mathbf{N} = -\mathbf{k}$ and

$$\mathbf{F}(x, y, 0) = e^{-y^2}\mathbf{i} - \mathbf{k}. \quad \text{so} \quad \mathbf{F} \cdot \mathbf{N} dA = dudv.$$

Thus if $D$ is the unit disk in the $(u, v)$-plane,

$$\int\int_{\mathbf{S_1}} \mathbf{F} \cdot \mathbf{N} dA = \int\int_D dudv = \pi.$$

c. Use the divergence theorem to evaluate the flux integral

$$\int\int_{\mathbf{S_2}} \mathbf{F} \cdot \mathbf{N} dA = \int\int_{\mathbf{S_2}} \mathbf{F} \cdot \mathbf{dS},$$

where $\mathbf{S_2}$ is the hemisphere

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

and $\mathbf{N}$ is the outward-pointing unit normal. (Hint: $\mathbf{S_1} + \mathbf{S_2}$ bounds a region in $(x, y, z)$-space.)

Partial solution: If $E$ is the region in $\mathbb{R}^3$ bounded by $\mathbf{S_1} + \mathbf{S_2}$, then it follows from the divergence theorem that

$$\int\int_{\mathbf{S_1}} \mathbf{F} \cdot \mathbf{N} dA + \int\int_{\mathbf{S_2}} \mathbf{F} \cdot \mathbf{N} dA = \int\int\int_E (\nabla \cdot \mathbf{F}) dxdydz = 0,$$

so

$$\int\int_{\mathbf{S_2}} \mathbf{F} \cdot \mathbf{N} dA = -\int\int_{\mathbf{S_1}} \mathbf{F} \cdot \mathbf{N} dA = -\pi.$$

3. Let $\mathbf{C}$ be the boundary of the part of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant $x \geq 0, y \geq 0, z \geq 0$, oriented to be traversed in the counterclockwise manner as viewed from above. Use Stokes' theorem to evaluate the line integral

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{F}(x, y, z) = (e^{-(x^2+1)}, x, \sqrt{z^4 + 1})$.

Partial solution: First we calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \partial/\partial x & e^{-(x^2+1)} \\ \mathbf{j} & \partial/\partial y & x \\ \mathbf{k} & \partial/\partial z & z^4 + 1 \end{vmatrix} = \mathbf{k}.$$

We next parametrize $\mathbf{S}$ by the map $\mathbf{x} : D \to \mathbb{R}^3$, where

$$D = \{(\phi, \theta) \in \mathbb{R}^2 : 0 \leq \phi \leq \pi/2, \ 0 \leq \theta \leq \pi/2\}$$

and

$$\mathbf{x}(\phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}.$$

Then

$$\frac{\partial \mathbf{x}}{\partial \phi}(\phi, \theta) = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix}, \qquad \frac{\partial \mathbf{x}}{\partial \theta}(\phi, \theta) = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix} = \sin\phi \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

Hence

$$\mathbf{N} dA = \pm \frac{\partial \mathbf{x}}{\partial \phi} \times \frac{\partial \mathbf{x}}{\partial \theta} d\phi d\theta = \cdots = \pm \sin\phi \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} d\phi d\theta.$$

The vector points up when $(\phi, \theta)$ in $D$, so we choose the plus sign in this expression. Then $\nabla \times \mathbf{F} \cdot \mathbf{N} dA = (\sin\phi\cos\phi) d\phi d\theta$, so

$$\begin{aligned} \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{x} &= \int\int_{\mathbf{S}} \nabla \times \mathbf{F} \cdot \mathbf{N} dA \\ &= \int_D (\sin\phi\cos\phi) d\phi d\theta = \int_0^{\pi/2} \left[ \int_0^{\pi/2} (\sin\phi\cos\phi) d\phi \right] d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} \sin^2\phi \Big|_0^{\pi/2} d\theta = \frac{\pi}{4}. \end{aligned}$$

4. a. Let $\mathbf{S}$ be the part of the paraboloid $z = x^2 - y^2$ which satisfies the inequalities $0 \le x \le 1$, $0 \le y \le 1$. Give a parametrization for $\mathbf{S}$.

Answer: We let

$$D = \{(u,v) \in \mathbb{R}^2 : 0 \le u \le 1,\ 0 \le v \le 1\}$$

and define $\mathbf{x} : D \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = \begin{pmatrix} u \\ v \\ u^2 - v^2 \end{pmatrix}.$$

b. Use Stokes' Theorem to evaluate the line integral

$$\int_{\partial \mathbf{S}} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial \mathbf{S}} \mathbf{F} \cdot \mathbf{T} ds, \qquad \text{where} \qquad \mathbf{F} = (-yz, xz, \cos(\log(z^2+5))),$$

where $\partial \mathbf{S}$ is the boundary of $\mathbf{S}$, oriented counterclockwise as viewed from above.

Partial solution: Using our parametrization, we find that

$$\frac{\partial \mathbf{x}}{\partial u}(u,v) = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \qquad \frac{\partial \mathbf{x}}{\partial v}(u,v) = \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix},$$

so

$$\mathbf{N} dA = \pm \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} dudv = \pm \left| \begin{pmatrix} \mathbf{i} & 1 & 0 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 2u & -2v \end{pmatrix} \right| dudv = \pm \begin{pmatrix} -2u \\ 2v \\ 1 \end{pmatrix} dudv.$$

In the application to Stokes's Theorem, we will want to take the upward-pointing unit normal, so we choose the plus sign in this last expression. Then since

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \partial/\partial x & -yz \\ \mathbf{j} & \partial/\partial y & xz \\ \mathbf{k} & \partial/\partial z & \cos(\log(z^2+5))) \end{vmatrix} = 2z\mathbf{k}.$$

we find that

$$\nabla \times \mathbf{F} \cdot \mathbf{N} dA = 2z dudv = 2(u^2 - v^2) dudv.$$

It now follows from Stokes' Theorem that

$$\int_{\partial \mathbf{S}} \mathbf{F} \cdot \mathbf{T} ds = \int \int_{\mathbf{S}} \nabla \times \mathbf{F} \cdot \mathbf{N} dA = \int \int_D 2(u^2 - v^2) dudv = \cdots = 0.$$

5. Let $\mathbf{S}$ be the oriented surface parametrized by $\mathbf{x} : D \to \mathbb{R}^3$,where

$$D = \{(u,v) \in \mathbb{R}^2 : -1 \le u \le 1, -1 \le v \le 1\}, \qquad \mathbf{x}(u,v) = (u, v, uv),$$

the orientation being the one determined by the parametrization. Let $\partial\mathbf{S}$ denote the oriented boundary of $\mathbf{S}$. Use Stokes' Theorem to evaluate the line integral

$$\int_{\partial\mathbf{S}} \mathbf{F}\cdot d\mathbf{x} = \int_{\partial\mathbf{S}} \mathbf{F}\cdot\mathbf{T}ds, \qquad \text{where} \qquad \mathbf{F} = (e^{-x^2}+12, x, \sin z).$$

Partial solution: First note that

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \partial/\partial x & e^{-x^2}+12 \\ \mathbf{j} & \partial/\partial y & x \\ \mathbf{k} & \partial/\partial z & \sin z \end{vmatrix} = \mathbf{k}.$$

Using the parametrization given for $\mathbf{S}$, we find that

$$\frac{\partial\mathbf{x}}{\partial u}(u,v) = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}, \qquad \frac{\partial\mathbf{x}}{\partial v}(u,v) = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix},$$

so

$$\mathbf{N}dA = \pm\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}dudv = \pm\left|\begin{pmatrix} \mathbf{i} & 1 & 0 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & v & u \end{pmatrix}\right| dudv = \pm\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} dudv.$$

To get the upward-pointing unit normal, we choose the plus sign. Then

$$(\nabla\times\mathbf{F})\cdot\mathbf{N}dA = dudv.$$

Hence by Stokes' Theorem,

$$\int_{\partial\mathbf{S}} \mathbf{F}\cdot\mathbf{T}ds = \int\int_{\mathbf{S}} (\nabla\times\mathbf{F})\cdot\mathbf{N}dA = \int_{-1}^{1}\int_{-1}^{1} dudv = 4.$$