Math 3CI: Practice Midterm

November 1, 2009

There will be two parts to the midterm, a take-home part and an in-class part. This Practice Midterm is for the in-class part of the midterm only.

1. The following populations and yearly growth rates come from Wikipedia and www.indexmundi.com:

Country	Population in 2009	Yearly growth rate	Rate constant k
Uganda	32,700,000	3.6%	.0354
Somalia	9,130,000	2.82%	.0278
Iraq	30,700,000	2.56%	.0253
Haiti	10,000,000	2.49%	.0246
Egypt	77,000,000	1.68%	.0167
India	1,170,000,000	1.58%	.0157
USA	308,000,000	.88%	.0088
China	1,333,000,000	.63%	.0063
UK	61,600,000	.28%	.0028
Japan	128,000,000	.14%	.0014
Germany	82,000,000	.04%	.0004

Assuming that the population growth rate remains constant and that $\log 2$ is approximately .7, about how long will it take for the population of Japan to double? About how long will it take the population of Uganda to double?

2. Let us consider the following modified version of the Volterra-Lotka predator-prey equations

$$\begin{aligned} \frac{dx}{dt} &= x - xy - .01x^2 \\ \frac{dy}{dt} &= -y + xy - .01y^2, \end{aligned} \tag{1}$$

Let h be a small real number, for example $h = .01$. Recalling that

$$\frac{x(t+h) - x(t)}{h} \quad \text{is an approximation to} \quad \frac{dx}{dt}(t)$$

and

$$\frac{y(t+h) - y(t)}{h} \quad \text{is an approximation to} \quad \frac{dy}{dt}(t)$$

write out a system of difference equations

$$x(t+h) = x(t) + h(\cdots), \qquad y(t+h) = y(t) + h(\cdots)$$

which should give approximations to the solutions to the system (1).

3. Sketch the solution curves $y = y(t)$ to the differential equation

$$\frac{dy}{dt} = -(y-1)^2$$

without solving the equation. Then solve the differential equation by separation of variables.

4. Use separation of variables to find the general solution to the differential equation

$$\frac{dy}{dt} = \pm\sqrt{4-y^2}.$$

5. Use variation of parameters to find the general solution to the differential equation

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1}{t^2}.$$

6. Use variation of parameters to find the general solution to the differential equation

$$(1+t)\frac{dy}{dt} + y = \cos t.$$

7. Use variation of parameters to find the general solution to the differential equation

$$\frac{dy}{dt} + 2ty = te^{-t^2}.$$

In the next few problems, we consider a cart moving along a friction-free track attached to a wall by means of a spring, with

$$x(t) = \text{position of the cart to the right of equilibrium at time } t.$$

We suppose that the cart is subject to a spring force and an external force $f(t)$. This leads to the differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t), \tag{2}$$

where the constants m and k are positive. Note that this is a second-order linear differential equation, nonhomogeneous if $f(t)$ is nonzero.

8. Suppose that $m = 1$, $k = 4$ and $f(t) = 0$. Then (2) becomes

$$\frac{d^2x}{dt^2} + 4x = 0, \tag{3}$$

a special case of the equation of *simple harmonic motion*. What is the general solution to this equation? (Hint: Try using $x = \cos(at)$ or $x = \sin(at)$ for some choice of a.)

9. What is the solution to the initial value problem

$$\frac{d^2x}{dt^2} + 4x = 0, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0?$$

10. Find a particular solution to the forced vibration problem

$$\frac{d^2x}{dt^2} + 4x = \sin(\omega t),$$

where $\omega \neq 2$. What is the amplitude $A(\omega)$ to the forced vibration as a function of ω? Graph $A(\omega)$ as a function of ω.

11. Find the general solution to

$$\frac{d^2x}{dt^2} + 4x = \sin(\omega t).$$

12. Find the solution to the initial value problem

$$\frac{d^2x}{dt^2} + 4x = \sin(t), \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$$

13. Find the constant solutions to the differential system

$$\begin{aligned} \tfrac{dx}{dt} &= y \\ \tfrac{dy}{dt} &= -2\sin x - 3y. \end{aligned} \tag{4}$$

If we approximate $\sin x$ by x we obtain the linearization of the system at $(x, y) = (0, 0)$. What is the general solution to the linearization? Is $(0, 0)$ a stable constant solution to the linearization? Can you draw any conclusion about the stability of the constant solution $(0, 0)$ to the original system? EXTRA CREDIT: Determine whether $(\pi, 0)$ is a stable constant solution.