Math 241B Winter 2016: Partial Differential Equations and Topology of Four-dimensional Manifolds

December 29, 2015

This course will meet in HSSB 4202, MWF 9-9:50.

The terminology used in the following outline will be defined and explained as the course progresses. Lecture notes will be provided on gauchospace.

1 Outline of Course

During the 1980's, Simon Donaldson used the Yang-Mills equations, which had originated in mathematical physics, to study the differential topology of four-manifolds, and obtained theorems of the following type:

Theorem A. *There exist many compact topological four-manifolds which have no smooth structure. In fact, the only compact simply connected smooth four-manifolds which have definite intersection forms are those whose intersection forms are represented by* $\pm I$*, where* I *is the identity matrix.*

Theorem B. *There exist many pairs of compact simply connected smooth four-manifolds which are homeomorphic but not diffeomorphic. In fact, some compact simply connected topological four-manifolds have infinitely many distinct smooth structures.*

For example, in the case where M is an algebraic hypersurface in $P^3\mathbb{C}$ of degree $d = 5$, it was shown that M is homeomorphic but not diffeomorphic to

$$\overbrace{P^2\mathbb{C}\sharp\cdots\sharp P^2\mathbb{C}}^{9}\,\sharp\,\overbrace{\overline{P^2\mathbb{C}}\sharp\cdots\sharp\overline{P^2\mathbb{C}}}^{44}.$$

Here $\sharp$ denotes the connected sum operation, which will be described within the course.

The proofs of Donaldson were based upon the Yang-Mills equations for connections on $SU(2)$-bundles over the four-manifold M. The arguments were lengthy and required substantial expertise in the theory of nonlinear partial differential equations.

In the fall of 1994, Edward Witten proposed a set of equations (developed jointly with Nathan Seiberg) which give the main results of Donaldson theory with easier arguments than had been thought possible. The Seiberg-Witten equations were less nonlinear because they involved connections on $U(1)$-bundles and $U(1)$ is abelian, and this simplified the analysis underlying the arguments. Moreover, Witten presented physical arguments which suggested that there should be a strong relationship between solutions to the Yang-Mills equations and the Seiberg-Witten equations.

The Seiberg-Witten theory also provided new obstructions to the existence of metrics of positive scalar curvature on smooth four-manifolds, obstructions that depend upon the smooth structure and not just the homeomorphism class of the manifold.

This course will be concerned with developing the techniques needed for the Seiberg-Witten equations, but also for closely related nonlinear partial differential equations which are important in geometry, such as the equations for pseudoholomorphic curves used in the mathematical foundation for classical mechanics, and the theory of parametrized minimal surfaces.

We will begin with the theory of vector bundles, including the theory of connections and curvature, characteristic classes, construction of the so-called Thom form and classification of vector bundles.

We then meet the simplest nontrivial version of the Atiyah-Singer index theorem, known as the Riemann-Roch theorem, which is fundamental for the theory of holomorphic vector bundles over a Riemann surface. Following Atiyah and Bott, we will describe how the theory of connections leads to a classification of holomorphic structures on vector bundles of rank two over Riemann surfaces.

The key topic for the winter quarter course will be an introduction to Kaehler geometry. Kaehler manifolds provide examples of Riemannian manifolds for which curvature is more accessible than in the general case. Once Kaehler manifolds are introduced, we will describe the Riemann-Roch theorem in Kaehler geometry as a special case of the Atiyah-Singer index theorem. We will present the theorem of Kodaira which gives conditions under which a Kaehler manifold imbeds in Euclidean space. Finally, we will describe the beautiful classification of Kaehler manifolds of complex dimension two due to Enriques and Kodaira, together with the topological invariants of these four-manifolds.

In the spring quarter, we plan to treat the Seiberg-Witten equations, using the geometry of spin and spin-c structures. This will give us an opportunity to provide proofs of Theorems A and B.