Math. 5B Friday, April 15, 2011

READ LOVRIC, § 4.2.

Examples of Taylor series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \underbrace{R_2}_{\text{remainder term}} \qquad \text{where } |R_2| \le C|x|^3$$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2!}f''(x_0)h^2 + R_2$$

If $f(x) = e^x$, then $f'(x) = f''(x) = \cdots = e^x$

$$f(x_0+h) = e^{x_0} + e^{x_0}\cdot h + \frac{1}{2!}e^{x_0}h^2 + R_2$$

If $x = x_0 + h$ $\quad h = x - x_0$

$$f(x) = \underbrace{\underbrace{e^{x_0} + e^{x_0}(x-x_0)}_{l(x)\ =\ \text{linear approximation}} + \frac{1}{2!}e^{x_0}(x-x_0)^2}_{q(x)\ =\ \text{quadratic approximation for } f(x) \text{ near } x_0} + R_2$$

Suppose now that $f(x,y) = 3x^2 + 3y^2 - 2x^3$.

$$f_x = \frac{\partial f}{\partial x} = 6x - 6x^2$$

$$f_y = \frac{\partial f}{\partial y} = 6y$$

What are critical points of f?

$$\begin{cases} 6x - 6x^2 = 0 \\ 6y = 0 \end{cases} \qquad \begin{cases} x(1-x) = 0 \\ y = 0 \end{cases} \qquad \begin{matrix} x = 0 \text{ OR } 1 \\ y = 0 \end{matrix}$$

$\boxed{(0,0) \text{ and } (1,0)}$

Hessian matrix = $\begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

THEOREM: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

In our example

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6-12x & 0 \\ 0 & 6 \end{pmatrix} \qquad \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

SECOND ORDER TAYLOR APPROXIMATION:

$$f(x_0+h, y_0+k) = f(x_0,y_0) + (f_x \;\; f_y)(x_0,y_0)\begin{pmatrix} h \\ k \end{pmatrix}$$
$$+ \frac{1}{2}(h \;\; k)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(x_0,y_0)\begin{pmatrix} h \\ k \end{pmatrix} + R_2$$

remainder term

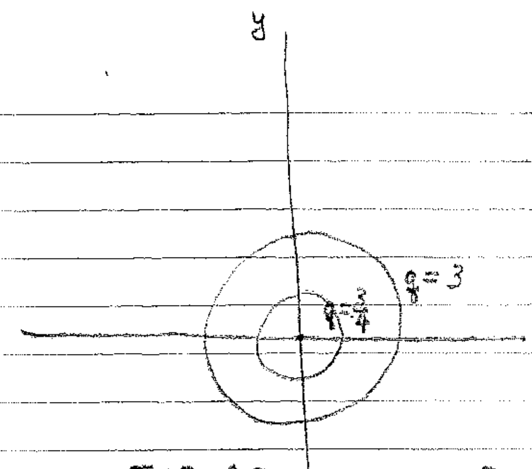
$x = x_0 + h \quad y = y_0 + k \qquad h = x - x_0\,, \; k = y - y_0$

$$f(x,y) = f(x_0,y_0) + (f_x \;\; f_y)(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$
$$+ \frac{1}{2}(x-x_0, y-y_0)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yy} & f_{yy} \end{pmatrix}(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + R_2$$

$f(x,y) = 3x^2 + 3y^2 - 2x^3 \qquad (x_0, y_0) = (0,0)$

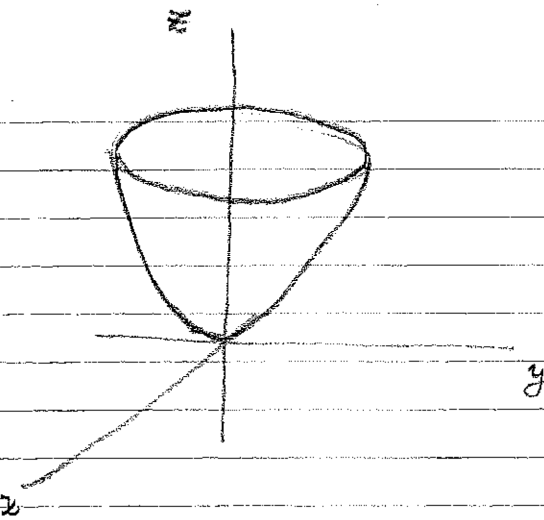
$$f(x,y) = (x \;\; y)\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + R_2 = 3x^2 + 3y^2 + R_2.$$

Near $(0,0)$, $f(x,y)$ is closely approximated by $q(x,y) = 3x^2 + 3y^2$



TOPOGRAPHIC MAP

level sets of q



$z = q(x,y) =$ osculating paraboloid.

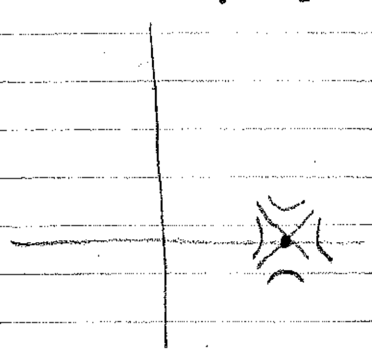
NEXT SUPPOSE $(x_0, y_0) = (1,0)$. THEN

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}$$

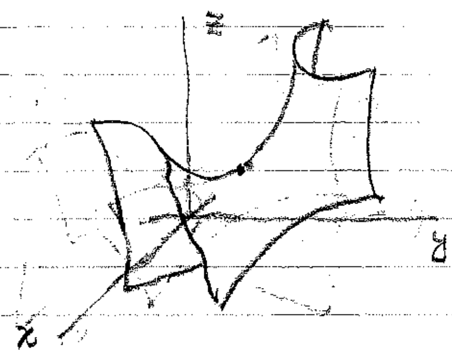
$$f(x,y) = f(1,0) + (x-1 \;\; y) \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} + R_2$$

$$= 1 - 3(x-1)^2 + 3y^2 + R_2$$

$$= q(x,y) + R_2 \quad \text{where} \quad q(x,y) = 1 - 3(x-1)^2 + 3y^2$$



LEVEL SETS OF q



$z = q(x,y) =$ osculating paraboloid

$(x_0, y_0) = (1,0)$ is a mountain pass (saddle point)