Math. 5B   Wednesday, April 13, 2011

Last time we considered the coordinate change

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

According to the CHAIN RULE

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix}$$

$$\begin{cases} \frac{dx}{dt} = \cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt} \\ \frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt} \end{cases}$$

$$\left(\frac{dx}{dt}\right)^2 = \cos^2\theta \left(\frac{dr}{dt}\right)^2 - 2r\sin\theta\cos\theta \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$\left(\frac{dy}{dt}\right)^2 = \sin^2\theta \left(\frac{dr}{dt}\right)^2 + 2r\sin\theta\cos\theta \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\cos^2\theta + \sin^2\theta)\left(\frac{dr}{dt}\right)^2 + r^2(\cos^2\theta + \sin^2\theta)\left(\frac{d\theta}{dt}\right)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

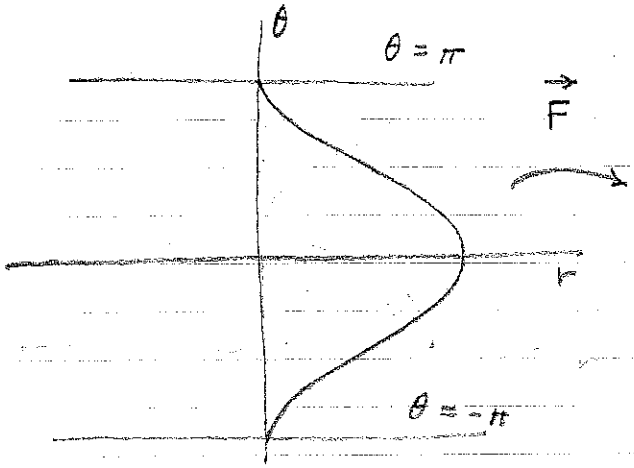
Suppose that $\vec{x} : [a,b] \longrightarrow \mathbb{R}^2$ is a parametrization of a curve C

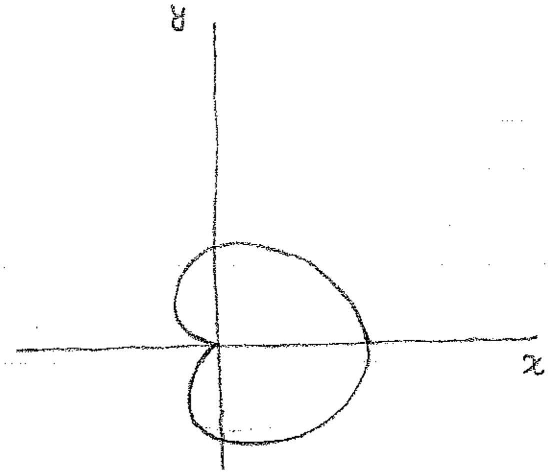
$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \text{Length of } C = \int_a^b |\vec{x}'(t)|\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: C given in polar coordinates, say

$r = 1 + \cos\theta, \quad -\pi < \theta < \pi$





CARDIOID

$$\text{Length of } C = \int_0^{2\pi} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt$$

$r(t) = 1 + \cos t$

$\theta(t) = t$

$\frac{dr}{dt} = -\sin t \qquad \frac{d\theta}{dt} = 1 \qquad \left(\frac{dr}{dt}\right)^2 = \sin^2 t$

$r^2\left(\frac{d\theta}{dt}\right)^2 = (1+\cos t)^2 = 1 + 2\cos t + \cos^2 t$

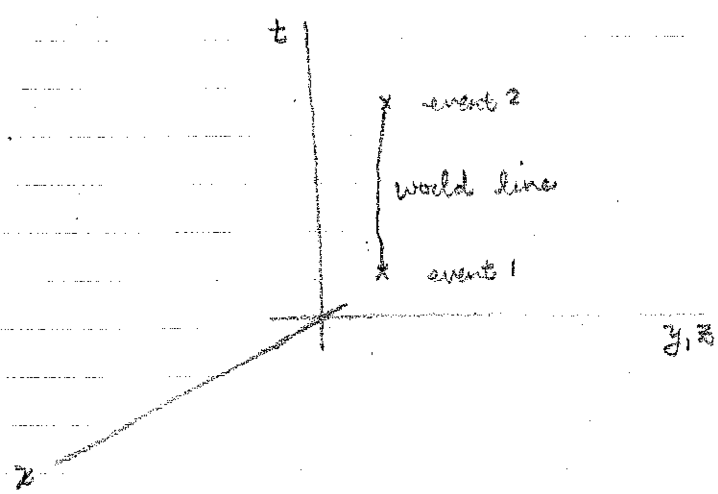
$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \sin^2 t + 1 + 2\cos t + \cos^2 t = 2 + 2\cos t$

$= 4\cos^2\left(\frac{t}{2}\right)$

↑ TRIG IDENTITY

$$L(C) = \int_{-\pi}^{\pi} 2\cos\left(\tfrac{t}{2}\right) dt = \int_{-\pi}^{\pi} 4\cos\left(\tfrac{t}{2}\right) d\left(\tfrac{t}{2}\right) = 4\sin\left(\tfrac{t}{2}\right)\Big|_{-\pi}^{\pi} = \boxed{8}.$$

SPECIAL RELATIVITY IN A NUTSHELL.



NOTHING CAN GO FASTER THAN LIGHT

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \leq c$$

c = speed of light

$c^2 dt^2 - dx^2 - dy^2 - dz^2 \geq 0$ along any world line

Change in time measured by an individual moving along a world line is ds, where

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \qquad t = \text{coordinate time}$$

Suppose a world line C is parametized by τ

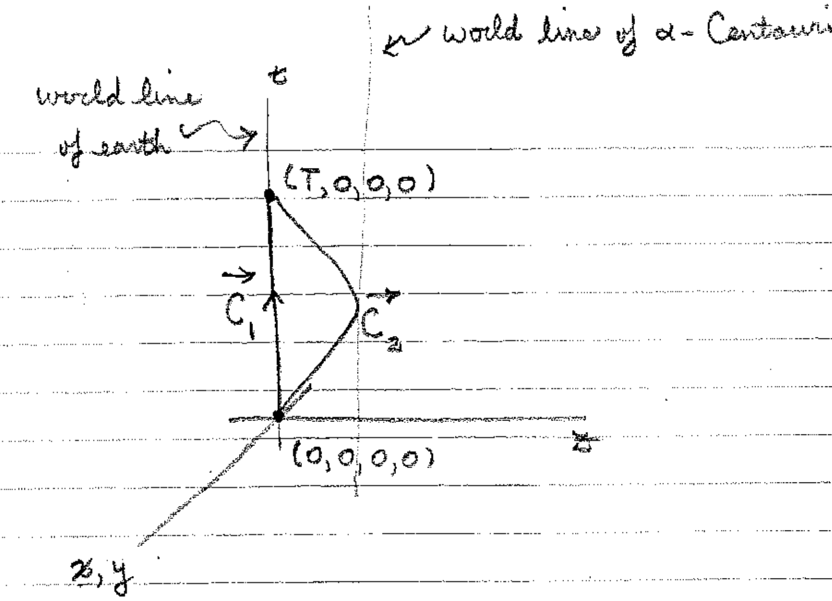
$$\vec{x}(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix} \qquad a \leq \tau \leq b.$$

Elapsed time along C between $\vec{x}(a)$ and $\vec{x}(b)$ is

$$L(C) = \int_a^b \frac{1}{c}\sqrt{c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau$$

KEY FACT:

L(C) is independent of parametization.



$$\vec{C}_1 : \begin{cases} t(\tau) = \tau \\ x(\tau) = 0 \\ y(\tau) = 0 \\ z(\tau) = 0 \end{cases} \qquad \vec{C}_2 \begin{cases} t(\tau) = \tau \\ x(\tau) = 0 \\ y(\tau) = 0 \\ z(\tau) = \begin{cases} c\tau & 0 \le \tau \le \frac{T}{2} \\ c(T-\tau) & \frac{T}{2} \le \tau \le T \end{cases} \end{cases}$$

$$L(\vec{C}_1) = \int_0^T \frac{1}{c}\sqrt{c^2}\, dt = T$$

$$L(\vec{C}_2) = \int_0^T \frac{1}{c}\sqrt{c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2} \underset{\text{EXERCISE}}{=} \cdots = 0$$

TWIN PARADOX: Elapsed time $L(\vec{C}_1)$ along $\vec{C}_1$ is NOT always equal to elapsed time $L(\vec{C}_2)$ along $\vec{C}_2$.

SPECIAL CASE: One could travel to α-Centauri (4.3 light years away) at (almost) the speed of light and then return at (almost) the speed of light (along $\vec{C}_2$).

A stationary observer on $\vec{C}_1$ would measure 8.6 years of elapsed time, while an observer moving along $\vec{C}_2$ would measure (almost) no time elapsed!