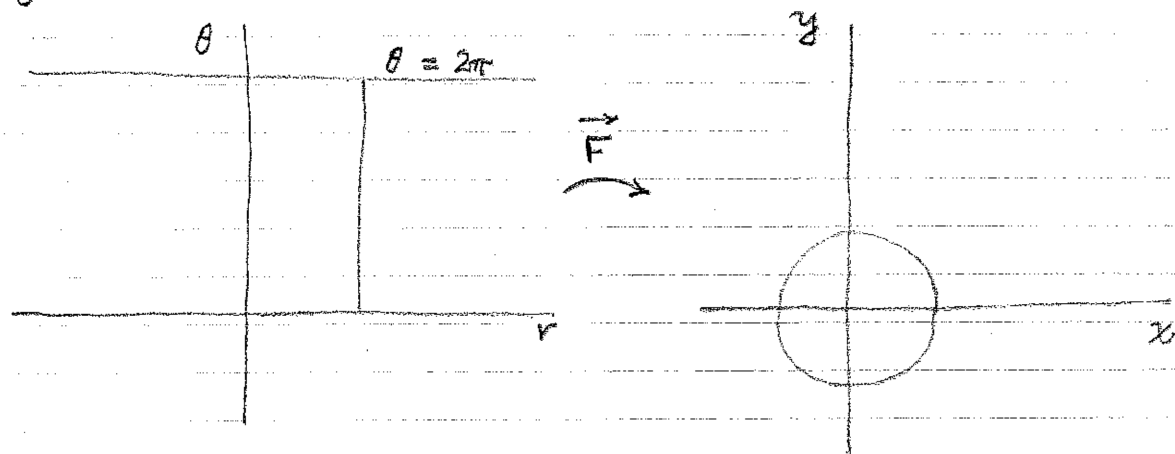
Math 5B. Monday, April 11, 2011

APPLICATION OF CHAIN RULE:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{F}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



$$\vec{G}(t) = \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix}$$

$$\vec{F}\circ\vec{G}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

CHAIN RULE:

$$\begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \end{pmatrix} = D(\vec{F}\circ\vec{G})(t) = D\vec{F}(\vec{G}(t))\cdot D\vec{G}(t)$$

$$= D\vec{F}(\vec{G}(t))\begin{pmatrix} \frac{dr}{dt}(t) \\ \frac{d\theta}{dt}(t) \end{pmatrix}$$

$$D\vec{F} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix}.$$

APPLICATION

$$\begin{cases} \dfrac{dx}{dt} = -4y \\ \dfrac{dy}{dt} = 4x \end{cases}$$

CAN WE SIMPLIFY BY CHANGING TO POLAR COORDINATES?

$$\begin{cases} \cos\theta \dfrac{dr}{dt} - r\sin\theta \dfrac{d\theta}{dt} = -4r\sin\theta & \times \cos\theta \\ \sin\theta \dfrac{dr}{dt} + r\cos\theta \dfrac{d\theta}{dt} = 4r\cos\theta & \times \sin\theta \end{cases}$$

$$(\cos^2\theta + \sin^2\theta)\frac{dr}{dt} = 0$$

$$\frac{dr}{dt} = 0$$

$$\begin{cases} -r\sin\theta \dfrac{d\theta}{dt} = -4r\sin\theta \\ r\cos\theta \dfrac{d\theta}{dt} = 4r\cos\theta \end{cases} \qquad \frac{d\theta}{dt} = 4$$

$$\begin{cases} \dfrac{dr}{dt} = 0 \\ \dfrac{d\theta}{dt} = 4 \end{cases} \qquad \begin{cases} r = a \\ \theta = 4t + b \end{cases}$$

$$\begin{cases} x = a\cos(4t+b) \\ y = a\sin(4t+b) \end{cases}$$

$$\begin{cases} m\dfrac{d^2x}{dt^2} = -\dfrac{GMm}{r^2}\dfrac{x}{r} \\ m\dfrac{d^2y}{dt^2} = -\dfrac{GMm}{r^2}\dfrac{y}{r} \end{cases}$$ ← NEWTON'S EQN FOR PLANETARY MOTION

(mass)(acceleration) = gravitational force

Easier to solve in polar coordinates.

$\gamma : [a,b] \to \mathbb{R}^3$ represents a parametrized curve

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \text{position of a moving particle in } \mathbb{R}^3$$

image of γ is an unparametrized curve in $\mathbb{R}^3$

$$\gamma'(t) = \begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \\ \frac{dz}{dt}(t) \end{pmatrix} = \text{velocity of particle at time } t.$$

$$|\gamma'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

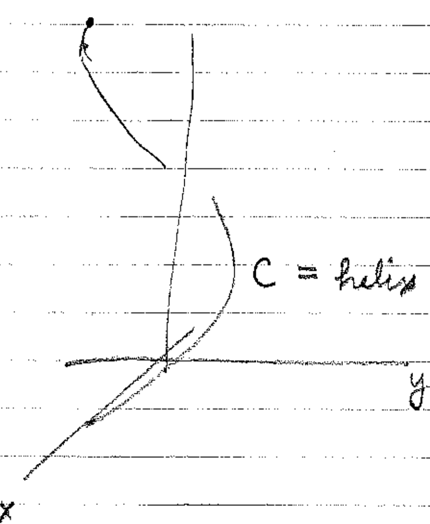
$$\int_a^b |\gamma'(t)|\, dt = \int_a^b (\text{speed})\, d(\text{time}) = \text{distance travelled}$$

If $\gamma : [a,b] \to \mathbb{R}^3$ is a parametrization of a curve C,

$$L(\gamma) = \text{length of } C = \int_a^b |\gamma'(t)|\, dt$$

EXAMPLE:

$$\gamma : [0, 2\pi] \to \mathbb{R}^3 \qquad \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$



$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

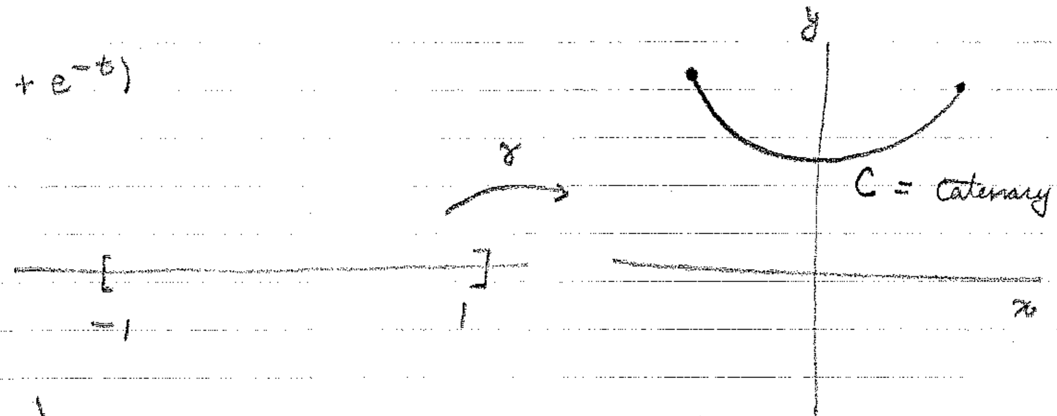
$$|\gamma'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \boxed{2\pi\sqrt{2}}$$

$$\gamma : [-1, 1] \longrightarrow \mathbb{R}^2 , \quad \gamma(t) = \begin{pmatrix} t \\ \cosh t \end{pmatrix}$$

$$\cosh t = \tfrac{1}{2}(e^t + e^{-t})$$



$$\gamma'(t) = \begin{pmatrix} 1 \\ \sinh t \end{pmatrix} \qquad |\gamma'(t)| = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t}$$

$$= \cosh t$$

$$L(\gamma) = \int_{-1}^{1} |\gamma'(t)|\,dt = \int_{-1}^{1} \cosh t\,dt = \sinh t \Big|_{-1}^{1} = \sinh 1 - \sinh(-1)$$

$$= 2\sinh 1 = e^1 - e^{-1} = \boxed{e - \tfrac{1}{e}.}$$