Math 5B   Friday, April 8, 2011.

If $f(x) = \sin x$ and $g(t) = e^t$ then $(f\circ g)(t) = \sin(e^t)$

CHAIN RULE: $(f\circ g)'(t) = f'(g(t))g'(t)$.

Suppose $z = f(x)$, $x = g(t)$. Then it is customary to write the chain rule as

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$$

$$\vec{F}: \begin{cases} z_1 = f_1(x_1,\dots,x_n) \\ \vdots \\ z_m = f_m(x_1,\dots x_n) \end{cases} \qquad \vec{G}: \begin{cases} x_1 = g_1(t_1,\dots,t_p) \\ \vdots \\ x_n = g_n(t_1,\dots,t_p) \end{cases}$$

CHAIN RULE VIA PARTIAL DERIVATIVES:

$$\frac{\partial z_i}{\partial t_j} = \frac{\partial z_i}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial z_i}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

CAN BE WRITTEN IN MATRICES

$$\begin{pmatrix} \frac{\partial z_1}{\partial t_1} & \cdots & \frac{\partial z_1}{\partial t_p} \\ & & \vdots \\ \frac{\partial z_m}{\partial t_1} & \cdots & \frac{\partial z_m}{\partial t_p} \end{pmatrix} = \begin{pmatrix} \frac{\partial z_1}{\partial x^1} & \cdots & \frac{\partial z_1}{\partial x^n} \\ & \vdots & \\ \frac{\partial z_m}{\partial x^1} & \cdots & \frac{\partial z_m}{\partial x^n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_p} \\ & & \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_p} \end{pmatrix}$$

$\vec{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \vec{G}: \mathbb{R}^m \longrightarrow \mathbb{R}^p$

$$\vec{F}\circ\vec{G}\begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix} = \vec{F}\left(\vec{G}\begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix}\right)$$

matrix mult.

CHAIN RULE $D(\vec{F}\circ\vec{G})(\vec{c}) = D\vec{F}(\vec{G}(\vec{c}))\, D\vec{G}(\vec{c})$

This is proving by reducing to the special case

$$z = f(x_1, \dots, x_n), \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{G}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Chain rule becomes

$$\frac{dz}{dt} = \left(\frac{\partial f}{\partial x^1} \cdots \frac{\partial f}{\partial x^n}\right)\begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

$$= \nabla f(\vec{x}(t)) \cdot \vec{x}'(t) \quad \text{where } \vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Think of $z = f(x,y)$ as temperature at (x,y) $\qquad x = x(t),\ y(t)$

$\vec{x}(t) = (x(t), y(t))$

$$\frac{dz}{dt}(t) = \nabla f(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}(t)$$

Rate of change of temperature $= (\nabla f)\cdot(\text{velocity})$.

Another application

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{F}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad \vec{G}(t) = \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix}$$

$$F \circ G(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$DF(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \qquad DG(t) = \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = D(F \circ G)(t) = DF(r(t), \theta(t))\, DG(t)$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix}$$

$$= \begin{matrix} \cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt} \\ \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt} \end{matrix}$$

SYSTEM OF ODE's:

$$\begin{cases} \frac{dx}{dt} = 4y \\ \frac{dy}{dt} = -4x \end{cases}$$

$$\begin{cases} \cos\theta \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt} = -4r\sin\theta & \times \cos\theta \\ \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt} = -4r\cos\theta & \times \sin\theta \end{cases}$$

$$1 \frac{dr}{dt} = 0 \qquad \frac{dr}{dt} = 0$$

$$\frac{d\theta}{dt} = -4$$

$$\begin{cases} \frac{dr}{dt} = 0 \\ \frac{d\theta}{dt} = -4 \end{cases} \longrightarrow \begin{cases} r = a \\ \theta = b - 4t \end{cases}$$