Math. 58 Wednesday, April 6, 2011

TODAY'S LECTURE CORRESPONDS TO § 2.4 AND § 2.7 IN LOVRIC.

Calculus of several variables concerns functions

$\vec{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

The simplest functions are linear functions, which are functions represented by matrices

$$\vec{F}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

But most functions are nonlinear.

Let us focus on the special case

$\vec{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

The derivative of $\vec{F}$ at (x_0, y_0) is a matrix

$$D\vec{F}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}.$$

EXAMPLE : $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{xy} \\ x^2 + y^2 \end{pmatrix}$

$$\vec{DF}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 2x & 2y \end{pmatrix}.$$

$$\vec{DF}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Last time we described how to linear $f(x,y)$ at (x_0, y_0)

Suppose we want to linearize

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \text{ at } (x_0, y_0).$$

Linearization of f at (x_0, y_0) is

$$l_1\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{\partial f}{\partial x}(x_0,y_0) \;\; \frac{\partial f}{\partial y}(x_0,y_0) \right) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + f(x_0,y_0)$$

Linearization of g at (x_0,y_0) is

$$l_2\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{\partial g}{\partial x}(x_0,y_0) \;\; \frac{\partial g}{\partial y}(x_0,y_0) \right) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + g(x_0,y_0)$$

Linearization of $\vec{F}$ at (x_0, y_0) is

$$\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix}$$

$$= \vec{DF}(x_0,y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \vec{F}(x_0,y_0)$$

$$\vec{L}(\vec{x}) = \vec{DF}(x_0)(\vec{x} - x_0) + \vec{F}(\vec{x}_0)$$

EXAMPLE

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y + x^2 \\ 4x + y + xy \end{pmatrix}$$

$$D\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+2x & 3 \\ 4+y & 1+x \end{pmatrix}$$

$$D\vec{F}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \qquad \vec{F}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x-0 \\ y-0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x+y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{F}(x,y) = \begin{pmatrix} \ln(x^2+y^2+1) \\ xy \end{pmatrix}$$

$$D\vec{F}(x,y) = \begin{pmatrix} \frac{2x}{x^2+y^2+1} & \frac{2y}{x^2+y^2+1} \\ y & x \end{pmatrix}$$

$$D\vec{F}(1,1) = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 1 & 1 \end{pmatrix} \qquad \vec{F}(1,1) = \begin{pmatrix} \ln 3 \\ 1 \end{pmatrix}$$

$$\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \begin{pmatrix} \ln 3 \\ 1 \end{pmatrix}.$$

PROBLEM FOR FRIDAY'S QUIZ:

Find the angle θ between the planes

Π_1: $2x + y + z = 3$

Π_2: $x + y = \pi$

SOLUTION:

$\vec{a} = (2, 1, 1)$ is $\perp$ to Π_1 (Do you see why?)

$\vec{b} = (1, 1, 0)$ is $\perp$ to Π_2

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2+1}{\sqrt{6}\,\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\theta = \frac{\pi}{6} = 30^\circ}$$

θ	$\cos\theta$	
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	NICE
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	TO KNOW!
$\frac{\pi}{3}$	$\frac{1}{2}$	