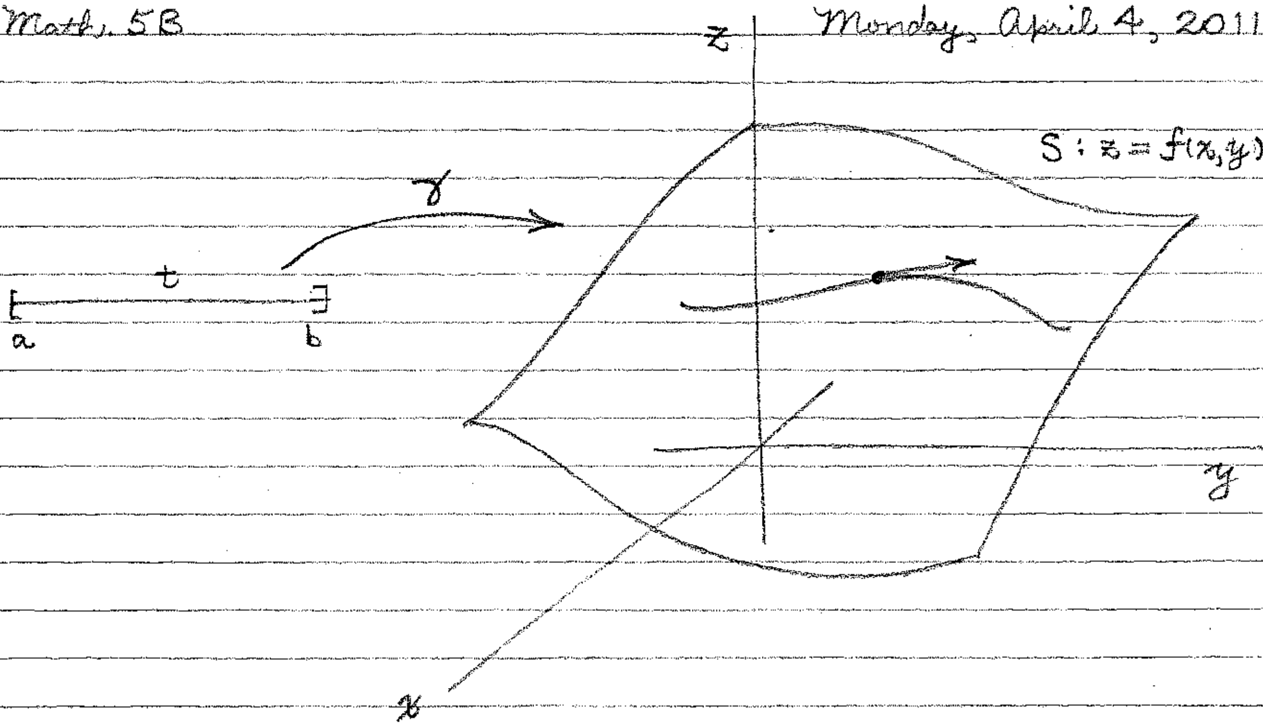
Math. 5B Monday, April 4, 2011



Suppose $\gamma(t) = (x(t), y(t), z(t))$ is a smooth curve

$t \in [a,b]$ lying on $S : z = f(x,y)$

Thus $z(t) = f(x(t), y(t))$

If $t_0 \in [a,b]$, $\gamma'(t_0) = \left(\frac{dx}{dt}(t_0), \frac{dy}{dt}(t_0), \frac{dz}{dt}(t_0)\right)$

is tangent to S.

$$\vec{N} = \begin{pmatrix} -\frac{\partial f}{\partial x}(x(t_0), y(t_0)) \\ -\frac{\partial f}{\partial y}(x(t_0), y(t_0)) \\ 1 \end{pmatrix}$$ is $\perp$ to S at $\gamma(t_0)$.

Hence

$$\begin{pmatrix} -\frac{\partial f}{\partial x}(x(t_0), y(t_0)) \\ -\frac{\partial f}{\partial y}(x(t_0), y(t_0)) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{dx}{dt}(t_0) \\ \frac{dy}{dt}(t_0) \\ \frac{dz}{dt}(t_0) \end{pmatrix} = 0$$

$$-\frac{\partial f}{\partial x}(x(t_0), y(t_0)) \cdot \frac{dx}{dt}(t_0) - \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \cdot \frac{dy}{dt}(t_0) + \frac{dz}{dt}(t_0) = 0$$

$$\boxed{\frac{dz}{dt}(t_0) = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \frac{dy}{dt}(t_0)}$$

This is the <u>chain rule</u>

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Application. The volume of a cylindrical can of radius r and height h is

$$V = f(r,h) = \pi r^2 h$$

If $r = 2$ ft $h = 5$ ft $\frac{dr}{dt} = 1$ ft/sec $\frac{dh}{dt} = 2$ ft/sec

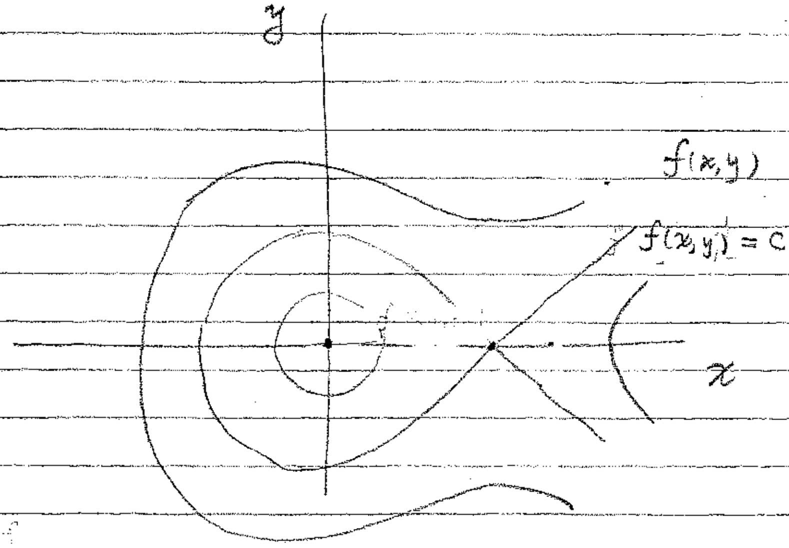
what is $\frac{dV}{dt}$?

<u>Chain rule</u>: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$= 2\pi \cdot 2 \cdot 5 \cdot 1 + \pi 2^2 \cdot 2 = 20\pi + 8\pi = 28\pi \text{ ft}^3/\text{s}$$

LEVEL SET VIEWPOINT:



$$f(x,y) = \frac{1}{4}x^2 + \frac{1}{4}y^2 - \frac{1}{6}x^3$$

Definition. The gradient of $f(x,y)$ at (x_0,y_0) is

$$\nabla f(x_0,y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) \end{pmatrix}$$

If we allow (x_0,y_0) we get a vector field ∇f on $\mathbb{R}^2$:

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix}.$$

Suppose $\vec{x} : [a,b] \longrightarrow \mathbb{R}^2$, $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a curve in $\mathbb{R}^3$ which PROJECTS to $\vec{x}(t)$.

Chain rule becomes

$$\frac{dz}{dt}(t_0) = \nabla f(x(t_0), y(t_0)) \cdot \frac{d\vec{x}}{dt}(t_0) \quad \text{where} \quad \frac{d\vec{x}}{dt}(t_0) = \begin{pmatrix} \frac{dx}{dt}(t_0) \\ \frac{dy}{dt}(t_0) \end{pmatrix}.$$

Suppose $f(x,y)$ = temperature at (x,y)

(rate of change of temperature with respect to t)

$= (\text{gradient of } f) \cdot (\text{velocity})$

$= (\nabla f) \cdot \vec{v}$

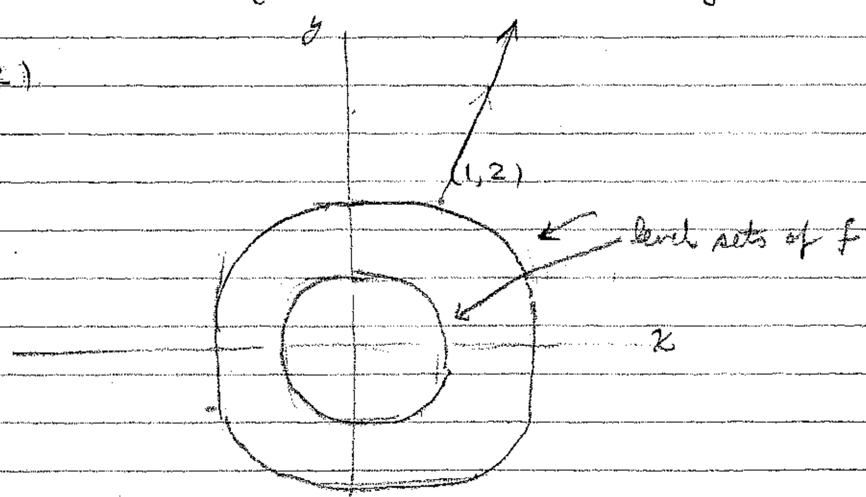
$= |\nabla f|\,|\vec{v}| \cos\theta$

where $\theta = \angle$ between ∇f and $|\vec{v}|$

If $\vec{v} = 1$, max rate of change occurs when $\vec{v}$ points in same direction as ∇f

∇f points in direction of maximum increase of f and $|\nabla f|$ is the rate of increase in this direction.

PROBLEM. Find a nonzero vector $\perp$ to $x^4 + y^4 = 17$ at the point $(1, 2)$



$f(x,y) = x^4 + y^4$ $\qquad \nabla f = 4x^3\vec{i} + 4y^3\vec{j}$

$\nabla f(1,2) = 4\vec{i} + 8\vec{j}$