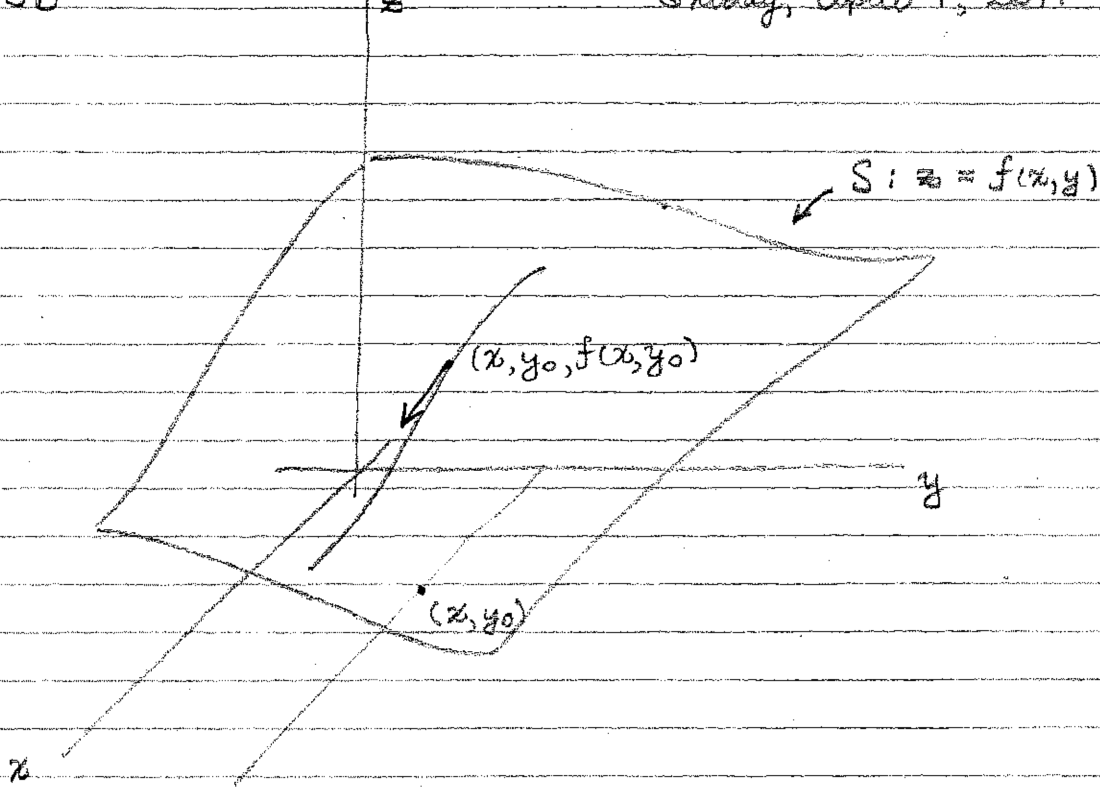
Math 5B Friday, April 1, 2011



$\gamma_1(x) = (x, y_0, f(x, y_0))$ is a curve on the surface S.

$\vec{v}_1 = \gamma_1'(x_0) = (1, 0, \frac{\partial f}{\partial x}(x_0, y_0))$ is tangent to S at $(x_0, y_0, f(x_0, y_0))$

$\gamma_2(y) = (x_0, y, f(x_0, y))$ is a curve on S.

$\vec{v}_2 = \gamma_2'(y_0) = (0, 1, \frac{\partial f}{\partial y}(x_0, y_0))$ is tangent to S $(x_0, y_0, f(x_0, y_0))$.

$$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0) \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(x_0, y_0)\,\vec{i} - \frac{\partial f}{\partial y}(x_0, y_0)\,\vec{j} + \vec{k}$$

is $\perp$ to S at $(x_0, y_0, f(x_0, y_0))$.

Tangent plane to S at $(x_0, y_0, f(x_0,y_0))$:

$$\vec{N} \perp (x - x_0,\ y - y_0,\ z - f(x_0,y_0))$$

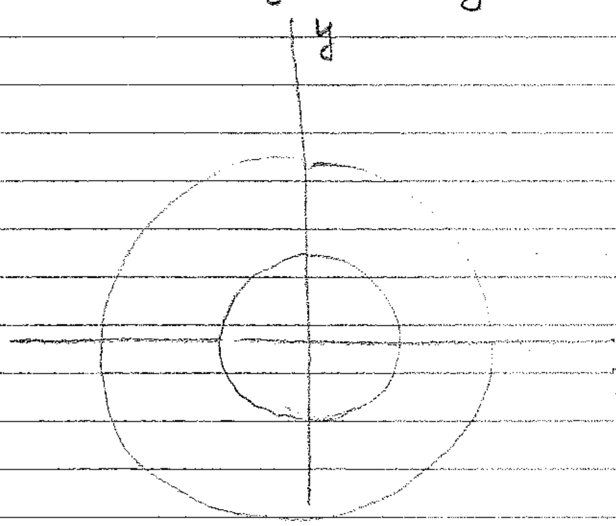
$$\vec{N} \cdot (x - x_0,\ y - y_0,\ z - f(x_0,y_0)) = 0$$

$$-\frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + z - f(x_0,y_0) = 0$$

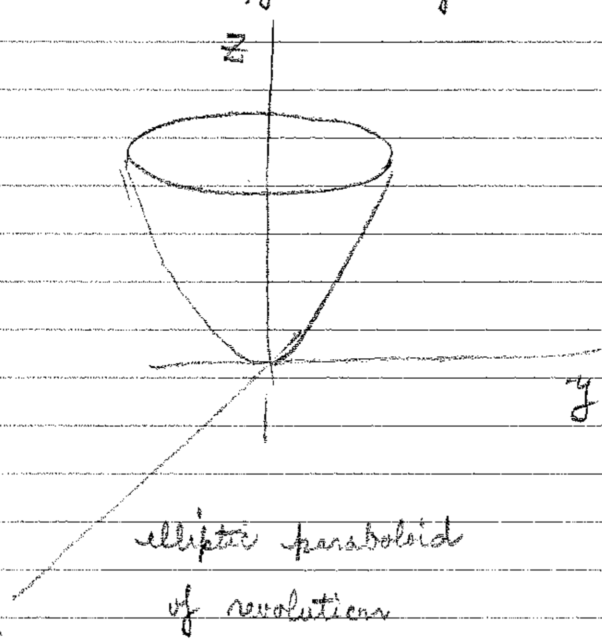
$$\boxed{z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0).}$$

Example:

$S:\ z = f(x,y) = x^2 + y^2$ level sets $f(x,y) = x^2 + y^2 = c$



topographic map



elliptic paraboloid of revolution

What is the tangent plane at $(1, 2, 5)$?

$x_0 = 1, \quad y_0 = 2 \qquad f(x_0,y_0) = 5$

$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$

$\frac{\partial f}{\partial x}(1,2) = 2 \qquad \frac{\partial f}{\partial y}(1,2) = 4$

Equation of tangent plane

$$\boxed{z = 5 + 2(x-1) + 4(y-2)}$$

$$z = l(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

is called the <u>linearization</u> of f at $(1,2)$.

<u>Definition.</u> (x_0,y_0) is a <u>critical point</u> for the function $f(x,y)$ if the tangent plane to $S: z = f(x,y)$ is horizontal at (x_0,y_0)

This happens $\Leftrightarrow$ $\frac{\partial f}{\partial x}(x_0,y_0) = 0 = \frac{\partial f}{\partial y}(x_0,y_0)$

Critical points are candidates for maxima and minima.

Example: Find critical points for

$f(x,y) = 3x^2 + 4xy + 3y^2 - 2x - 8y$

$$\frac{\partial f}{\partial x} = 6x + 4y - 2 = 0$$

$$\frac{\partial f}{\partial y} = 4x + 6y - 8 = 0$$

$$\begin{cases} 6x + 4y = 2 \\ 4x + 6y = 8 \end{cases}$$

$$x = \frac{\begin{vmatrix} 2 & 4 \\ 8 & 6 \end{vmatrix}}{\begin{vmatrix} 6 & 4 \\ 4 & 6 \end{vmatrix}} = \frac{12 - 32}{36 - 16} = \frac{-20}{20} = -1$$

$$y = \frac{\begin{vmatrix} 6 & 2 \\ 4 & 8 \end{vmatrix}}{\begin{vmatrix} 6 & 4 \\ 4 & 6 \end{vmatrix}} = \frac{48 - 8}{36 - 16} = \frac{40}{20} = 2$$

Critical point is $(-1, 2)$

Later we will see how to show this is a local minimum.

$$L_1 : \vec{x}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$L_2 : \vec{y}(u) = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + u \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

What is the distance between L_1 and L_2?

$f(t,u) = [\text{distance from } \vec{x}(t) \text{ to } \vec{y}(u)]^2$

$f(t,u) = (1+2t-1-2u)^2 + (-t-4)^2 + (5-u)^2$

$= (2t-2u)^2 + (t+4)^2 + (u-5)^2$

$= 4t^2 - 8tu + 4u^2 + t^2 + 8t + 16 + u^2 - 10u + 25$

$= 5t^2 - 8tu + 5u^2 + 8t - 10u + 41$

$\frac{\partial f}{\partial t} = 10t - 8u + 8 = 0$

$\frac{\partial f}{\partial u} = -8t + 10u - 10 = 0$

$t = 0 \;, \; u = 1$

$f(0,1) = 2^2 + 4^2 + 4^2 = 36$

distance from L_1 to L_2 = $\sqrt{36} = 6.$