Math. 240C.   Wednesday, June 1, 2011

EXTRA OFFICE HOURS IN MATH LAB.

SH 1607, Mon June 6, 6:30 - 7:30 pm

The gradient operator $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ defines 3 operators

$$(\text{functions}) \xrightarrow{\nabla} (\text{vector fields}) \xrightarrow{\nabla\times} (\text{vector fields}) \xrightarrow{\nabla\cdot} (\text{functions})$$

PROPERTIES:

$\nabla \times (\nabla f) = 0$

$\nabla \cdot (\nabla \times \vec{F}) = 0.$

In fact, we have the following theorems:

POINCARÉ LEMMA. If $\vec{F}(x,y,z)$ is a smooth vector field on all of $\mathbb{R}^3$, then

$\nabla \times \vec{F} = \vec{0} \Rightarrow \exists$ a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = \vec{F}$

$\nabla \cdot \vec{F} = \vec{0} \Rightarrow \exists$ a smooth vector field $\vec{A}$ on $\mathbb{R}^3$ such that $\nabla \times \vec{A} = \vec{F}$.

Henri Poincaré (1854 - 1912).

EXAMPLE: $\vec{F}(x,y) = 2xy\,\vec{i} + (x^2+z^2)\,\vec{j} + 2yz\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & 2xy \\ \vec{j} & \frac{\partial}{\partial y} & x^2+z^2 \\ \vec{k} & \frac{\partial}{\partial z} & 2yz \end{vmatrix} = (2z-2z)\vec{i} + (0-0)\vec{j} + (2x-2x)\vec{k} = \vec{0}$$

So $\exists\ f(x,y,z)$ such that $\nabla f = \vec{F}$.

That means:

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2+z^2 \qquad \frac{\partial f}{\partial z} = 2yz$$

$$\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y,z) = x^2y + g(y,z)$$

integrate treating y, z as constants

constants of integration can be arbitrary function of y and z.

$$\frac{\partial f}{\partial y} = x^2+z^2 \Rightarrow x^2 + \frac{\partial g}{\partial y}(y,z) = x^2+z^2 \Rightarrow \frac{\partial g}{\partial y}(y,z) = z^2$$

$$\Rightarrow g(y,z) = yz^2 + h(z)$$

$$\frac{\partial f}{\partial z} = 2yz \Rightarrow \frac{\partial g}{\partial z} = 2yz \Rightarrow 2yz + \frac{\partial h}{\partial z}(z) = 2yz \Rightarrow \frac{\partial h}{\partial z} = 0,$$

so $h = c$, where c is a constant.

So $f(x,y,z) = x^2y + g(y,z) = x^2y + yz^2 + h(z) = x^2y + yz^2 + c$

$\nabla(x^2y + yz^2 + c) = \vec{F}$.

POINCARÉ LEMMA not necessarily true when $\vec{F}$ is not smooth on all of $\mathbb{R}^3$.

Key Theorems of Vector Calculus:

FUNDAMENTAL THEOREM.

If $\vec{C}$ is a piecewise smooth curve from P to Q, and $f(x,y,z)$ is a smooth function on $\vec{C}$, then

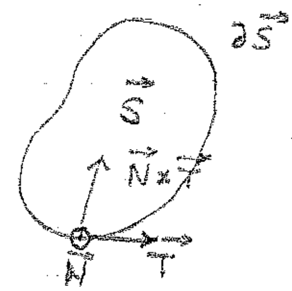
$$\int_{\vec{C}} \nabla f \cdot \vec{T}\, ds = f(Q) - f(P).$$

STOKES' THEOREM

If $\vec{S}$ is a smooth surface with piecewise smooth boundary $\partial\vec{S}$ and $\vec{F}(x,y,z)$ is smooth on S, then

$$\iint_{\vec{S}} (\nabla \times \vec{F}) \cdot \vec{N}\, dA = \int_{\partial\vec{S}} \vec{F} \cdot \vec{T}\, ds$$

when $\vec{N} \times \vec{T}$ points into $\vec{S}$



DIVERGENCE THEOREM.

If D is a region in $\mathbb{R}^3$ with piecewise smooth boundary ∂D and $\vec{F}(x,y,z)$ is smooth on D, then

$$\iiint_D \nabla \cdot \vec{F}\, dx\, dy\, dz = \iint_{\partial D} \vec{F} \cdot \vec{N}\, dA.$$

APPENDIX: POINCARÉ'S LEMMA NOT TRUE FOR VECTOR FIELDS WITH SINGULARITIES!

$$\vec{F}(x,y,z) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = (M(x,y), N(x,y), 0)$$

$\vec{F}$ is smooth on $\mathbb{R}^3 - \{z\text{-axis}\}$. $\vec{F}$ has a singularity on z-axis.

$$\frac{\partial M}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & M \\ \vec{j} & \frac{\partial}{\partial y} & N \\ \vec{k} & \frac{\partial}{\partial z} & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k} = 0\vec{k}.$$

But if $\vec{C}$ is the unit circle

$$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \qquad 0 \le t \le 2\pi,$$

$$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\vec{C}} \vec{F}\cdot d\vec{x} = \int_{\vec{C}} \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$$= \int_0^{2\pi} \frac{(-\sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)}{1}$$

$$= \int_0^{2\pi} dt = 2\pi.$$

If it were true that $\vec{F} = \nabla f$, then $\int_{\vec{C}} \vec{F}\cdot T\,ds = 0$ by FUNDAMENTAL THEOREM!