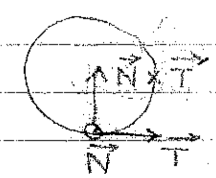
Math. 5B — Friday, May 27, 2011.

Stokes' Theorem.

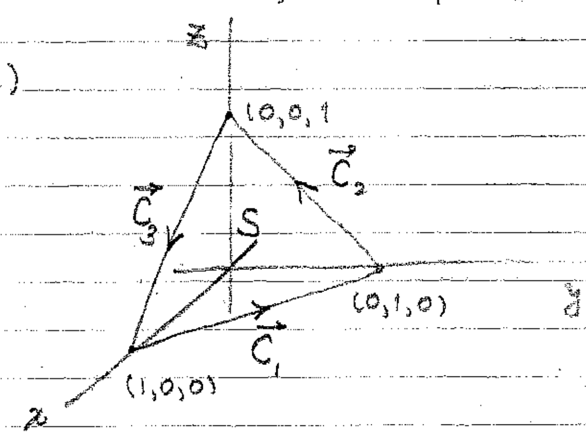
If $\vec{S}$ is a piecewise smooth surface in $\mathbb{R}^3$ bounded by a piecewise smooth curve $\partial\vec{S}$ and $\vec{F}$ is a smooth vector field on $\mathbb{R}^3$, then



$$\iint_S (\nabla\times\vec{F})\cdot\vec{N}\,dA = \int_{\partial S}\vec{F}\cdot\vec{T}\,ds$$

where $\vec{N}$ and $\vec{T}$ are chosen so that $\vec{N}\times\vec{T}$ points into $\vec{S}$.

EXAMPLE: Let $\vec{S}$ be the part the plane $x+y+z=1$ which lies in the octant $x\ge 0,\ y\ge 0,\ z\ge 0$. Let $\vec{C}=\partial\vec{S}$, $\vec{C}=\vec{C}_1+\vec{C}_2+\vec{C}_3$



If $\vec{F}(x,y,z) = (\log(x^2+1),\ z,\ -y)$

what is

$$\int_{\vec{C}}\vec{F}\cdot\vec{T}\,ds \ ?$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & \log(x^2+1) \\ \vec{j} & \frac{\partial}{\partial y} & z \\ k & \frac{\partial}{\partial z} & -y \end{vmatrix}$$

$$= -2i$$

$$\vec{S}:\ x+y+z=1 \qquad \vec{x}(u,v) = \begin{pmatrix} u \\ v \\ 1-u-v \end{pmatrix},\quad u+v\le 1\quad u\ge 0,\ v\ge 0$$

$$z = 1-x-y$$

$$\vec{x}: D \longrightarrow \mathbb{R}^2 \qquad D = \{(u,v):\ u\ge 0,\ v\ge 0,\ u+v\le 1\}$$

$$\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{N}\,dA = \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\,du\,dv = \begin{vmatrix} \vec{i} & 1 & 0 \\ \vec{j} & 0 & 1 \\ \vec{k} & -1 & -1 \end{vmatrix} du\,dv = (\vec{i} + \vec{j} + \vec{k})\,du\,dv$$

$$(\nabla \times \vec{F}) \cdot \vec{N}\,dA = (0,0,1) \cdot (1,1,1)\,du\,dv$$

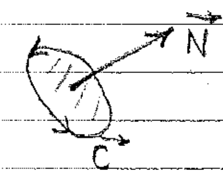
$$\iint_{\vec{S}} (\nabla \times \vec{F}) \cdot \vec{N}\,dA = \iint_D du\,dv = \int_0^1 \left[\int_0^{1-u} dv\,du\right]$$

$$= \int_0^1 (1-u)\,du = u - \tfrac{1}{2}u^2 \Big|_0^1 = \boxed{\tfrac{1}{2}}.$$

Interpretation of $\nabla \times \vec{F}$:

Suppose $\vec{S}$ is a disk in $\mathbb{R}^3$ with unit normal $\vec{N}$ and boundary $\vec{C}$

Then



$$\frac{1}{\text{Area}(\vec{S})} \int_C \vec{F} \cdot \vec{T}\,ds = \frac{1}{\text{Area}} \iint \nabla \times F \cdot \vec{N}\,dA$$

= Average circulation of $\vec{F}$ over S in direction of $\vec{N}$

$(\nabla \times \vec{F})(x_0, y_0, z_0) \cdot \vec{N}$ = Average circulation of $\vec{F}$ in dir. of $\vec{N}$ at (x_0, y_0, z_0)

Direction of $\nabla \times \vec{F}$ is direction which MAXIMIZES average circulation.

Maxwell's Equations (1861, 62):

$\vec{E}(x,y,z,t)$ = electric field $\qquad \rho(x,y,z,t)$ = charge density

$\vec{B}(x,y,z,t)$ = magnetic field $\qquad \vec{J}(x,y,z,t)$ = current density

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \qquad \nabla \times B - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

ε_0, μ_0, c constants

Time-independent case:

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss' Law) $\qquad \nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} = \mu_0 \vec{J}$ (Ampère's law) $\qquad \nabla \times \vec{E} = \vec{0}$

If $\vec{S}$ is a surface with normal $\vec{N}$, $\iint_{\vec{S}} \vec{J} \cdot \vec{N}\, dA$ = fluid passing thru S in direction of $\vec{N}$.

Stokes' Theorem $\iint_{\vec{S}} \vec{J} \cdot \vec{N}\, dA$

$= \frac{1}{\mu_0} \int_{\partial \vec{S}} \vec{B} \cdot \vec{T}\, ds.$

$= \int_{\partial \vec{S}} \mu_0 \vec{B} \cdot \vec{T}\, ds$

by Ampère's Law

EXAMPLE: Suppose that

$D = \{(x,y,z) : x^2+y^2 \leq \frac{1}{4}\}$ represents a wire on which a current flows with current density

$\frac{1}{\mu_0} \vec{B} = \frac{-y\,dx + x\,dy}{x^2+y^2}$ OUTSIDE D.

Current flowing along wire $= \iint_{\vec{S}} \vec{J} \cdot \vec{N}\, dA$

where $\vec{S} = \{(x,y,z) \in \mathbb{R} : z = 0 ; x^2+y^2 \leq 1\}$

What is $\iint_{\vec{S}} \vec{J} \cdot \vec{N}\, dA$, when $\vec{N}$ = upward-pointing unit normal.

$\partial \vec{S} = \vec{C} = \{(x,y,z) : z = 0, x^2+y^2 = 1\}$

$\iint_{\vec{S}} \vec{J} \cdot \vec{N}\, dA = \int_{\vec{C}} \frac{-y\,dx + x\,dy}{x^2+y^2}$

$$\begin{cases} x = \cos t \\ y = \sin t \qquad 0 \leq t \leq 2\pi \\ z = 0 \end{cases}$$

$dx = -\sin t \, dt$

$dy = \cos t \, dt$

$$\frac{-y\,dx + x\,dy}{x^2 + y^2} = \frac{(\sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)}{1}$$

$$= dt$$

$x^2 + y^2 = 1$

$$\iint_{\vec{S}} \vec{J} \cdot \vec{N} \, dA = \int_{\vec{C}} \frac{1}{\mu_0} \vec{B} \cdot \vec{T} \, ds = \int_{\vec{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^{2\pi} dt = \boxed{2\pi}.$$