Math. 5B. Wednesday, May 25, 2011.

DIVERGENCE THEOREM IN $\mathbb{R}^3$.

Let D be a region in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, $\vec{N}$ = outward-pointing unit normal to S. If $\vec{F}(x,y,z)$ is smooth on D and its boundary, then

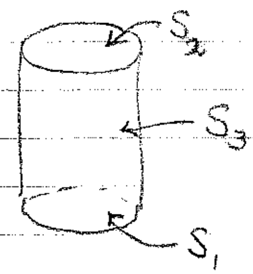
$$\iint_S \vec{F}\cdot\vec{N}\,dA = \iiint_D \nabla\cdot\vec{F}\,dx\,dy\,dz.$$

EXAMPLE: Find $\iint_S \vec{F}\cdot\vec{N}\,dA$, where

$S = \partial D$, $D = \{(x,y,z)\in\mathbb{R}^3 : 0 \le z \le 2,\ x^2+y^2\le 1\}$

$\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{N}$ = outward-pointing unit normal

Method I: $\partial D = S_1 + S_2 + S_3$



S_1: $\vec{x}(u,v) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$, $u^2+v^2\le 1$

$\vec{N} = -\vec{k}$, $\vec{F}\cdot\vec{N} = -z = 0$ $\iint_{S_1} \vec{F}\cdot\vec{N}\,dA = 0$

S_2: $x(u,v) = \begin{pmatrix} u \\ v \\ 2 \end{pmatrix}$, $u^2+v^2\le 1$

$\vec{N} = \vec{k}$, $\vec{F}\cdot\vec{N} = 2z = 2$ $\iint_{S_2} \vec{F}\cdot\vec{N}\,dA = \iint_{u^2+v^2\le 1} 2\,du\,dv$

$= 2\pi$

$$S_3 : \vec{x}(u,v) = \begin{pmatrix} \cos u \\ \sin u \\ v \end{pmatrix} \qquad 0 \le u \le 2\pi, \; 0 \le v \le 2$$

$$\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{N}\,dA = \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\, du\,dv = \begin{vmatrix} \vec{i} & -\sin u & 0 \\ \vec{j} & \cos u & 0 \\ \vec{k} & 0 & 1 \end{vmatrix} du\,dv = (\cos u, \sin u, 0)\, du\,dv$$

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \int_0^2 \left[\int_0^{2\pi} (\cos u, \sin u, v)\cdot(\cos u, \sin u, 0)\, du \right] dv$$

$$= \int_0^2 \left[\int_0^{2\pi} (\cos^2 u + \sin^2 u)\, du \right] dv = \int_0^2 \left[\int_0^{2\pi} du \right] dv = 4\pi$$

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \iint_{S_1} \vec{F}\cdot\vec{N}\,dA + \iint_{S_2} \vec{F}\cdot\vec{N}\,dA + \iint_{S_3} \vec{F}\cdot\vec{N}\,dA$$

$$= 0 + 2\pi + 4\pi = 6\pi$$

Method II: $\iint_S \vec{F}\cdot\vec{N}\,dA = \iiint_D \nabla\cdot\vec{F}\,dx\,dy\,dz = \iiint_D 3\,dx\,dy\,dz$

$$= 3 \text{ volume of } D = 6\pi.$$

IT IS IMPORTANT THAT $\vec{F}(x,y,z)$ BE SMOOTH OVER ALL OF D.

EXAMPLE: GAUSS' ELECTROSTATIC FIELD FOR A POINT PARTICE.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{(x^2+y^2+z^2)^{3/2}} (x\vec{i} + y\vec{j} + z\vec{k}) \quad \text{on } \mathbb{R}^3 - \{\vec{0}\}.$$

$$E = \nabla V, \text{ where } V = -\frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{x^2+y^2+z^2}}$$

$\nabla \cdot \vec{E} = 0$ at all points of $\mathbb{R}^3 - \{\vec{0}\}$.

See example 4.52 in Lorrie.

Suppose $S_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, $\vec{N}$ = outward pointing normal. $S_0 = \partial D_0$, where $D_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$

$$\iint_{S_0} \vec{E} \cdot \vec{N}\, dA = ?$$

$$S_0 : \begin{cases} x = \sin\phi \cos\theta \\ y = \sin\phi \sin\theta \\ z = \cos\phi \end{cases} \qquad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi$$

$$\vec{N}\, dA = \left(\frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \theta} \right) d\phi\, d\theta = \begin{vmatrix} \vec{i} & \cos\phi \cos\theta & -\sin\phi \sin\theta \\ \vec{j} & \cos\phi \sin\theta & \sin\phi \cos\theta \\ \vec{k} & -\sin\phi & 0 \end{vmatrix} d\phi\, d\theta$$

$$= \sin\phi \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} d\phi\, d\theta = \sin\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} d\phi\, d\theta$$

$$E \cdot \vec{N}\, dA = \frac{Q}{4\pi\varepsilon_0} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \sin\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} d\phi\, d\theta = \frac{Q}{4\pi\varepsilon_0} \sin\phi\, d\phi\, d\theta$$

$$\iint_{S_0} \vec{E} \cdot \vec{N}\, dA = \frac{Q}{4\pi\varepsilon_0} \int_0^{2\pi} \left[\int_0^{\pi} \sin\phi\, d\phi \right] d\theta = \frac{Q}{\varepsilon_0}$$

$$Q = \{\text{charge at origin}\} = \varepsilon_0 \iint_{S_0} \vec{E} \cdot \vec{N}\, dA$$

EVEN THOUGH $\nabla \cdot \vec{E} = 0$ on $\mathbb{R}^3 - \{0\}$ CANNOT APPLY DIV THM TO CONCLUDE THAT

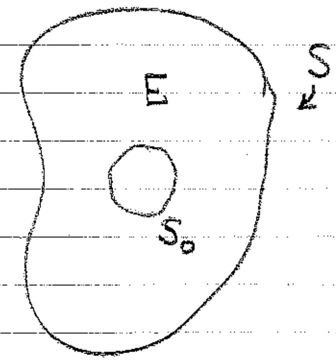
$$\iint_{S_0} \vec{E} \cdot \vec{N}\, dA = \iiint_{D_0} \nabla \cdot \vec{E}\, dx\, dy\, dz$$

BECAUSE $\vec{E}$ HAS A SINGULARITY IN D_0.

If $S = \partial D$, where $D \subseteq \mathbb{R}^3$ contains $\vec{0}$ in its interior

and $\vec{N}$ = outward-pointing normal

$$\iint_S E \cdot \vec{N}\, dA = \frac{Q}{\varepsilon_0}.$$



Indeed, if $S_0 \subseteq D$, we can apply DIV THM

To $E = D - D_0$

$$\iint_S \vec{N}\, dA + \iint_{S_0} (-\vec{N}\, dA) = \iiint_E \nabla \cdot \vec{E}\, dx\,dy\,dz = 0$$

$\vec{N}$ points outward in both cases

but $-\vec{N}$ points out of $E = D - D_0$

$$\iint_S \vec{N}\, dA = \iint_{S_0} \vec{N}\, dA = \frac{Q}{\varepsilon_0}.$$

Faraday and Maxwell imagined lines of force emanating from the charged particles.

But Maxwell's equations do not include point charges, but charge density,

$\rho(x,y,z)$ = charge per unit volume at (x,y,z).

ONE OF MAXWELL'S EQUATIONS: $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$

ρ is assumed to be well-behaved, so we can apply divergence theorem

If $S = \partial D$ and $\vec{N}$ = outward unit normal,

$$\iint_S \vec{E} \cdot \vec{N}\, dA = \iiint_D \nabla \cdot \vec{E}\, dx\,dy\,dz = \frac{1}{\varepsilon_0} \iiint_D \rho(x,y,z)\, dx\,dy\,dz.$$

Thus $\iint_S \vec{E} \cdot \vec{N} \, dA = \frac{1}{\varepsilon_0}$ (total charge within D).