Math. 5B Monday, May 23, 2011

<u>Green's Theorem.</u> Suppose that $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is smooth (with smooth partial derivatives) over a region D in (x,y)-plane bounded by a piecewise smooth curve $\vec{C}$ oriented so that so $\vec{C}$ is traversed in positive direction, D is on left. Then

$$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\vec{C}} \vec{F}\cdot d\vec{r} = \int_{\vec{C}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Example: $\vec{F}(x,y) = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

$M = -\frac{y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial M}{\partial y} = -\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ Let $D = \{(x,y) : x^2+y^2 \le 1\}$

Let $\vec{C}_1$ = unit circle $x^2+y^2=1$ traversed $\circlearrowleft$ $= \partial D$

Then $\int_{\vec{C}_1} M\,dx + N\,dy = ?$

$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $0 \le t \le \pi$ $\quad -\frac{y}{x^2+y^2} = -\sin t \quad dx = -\sin t\,dt$

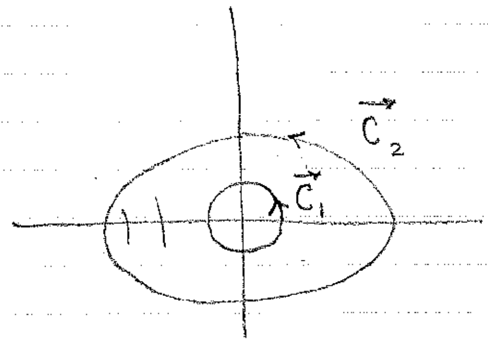
$\frac{x}{x^2+y^2} = \cos t \quad dy = \cos t\,dt$

$$\int_{\vec{C}_1} M\,dx + N\,dy = \int_0^{2\pi} (-\sin t)(-\sin t\,dt) + \cos^2 t\,dt = \int_0^{2\pi} dt = 2\pi$$

$$2\pi = \int_{\vec{C}_1} M\,dx + N\,dy \neq \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$$

THIS DOES NOT CONTRADICT GREEN'S THM BECAUSE M AND N ARE NOT DIFFERENTIABLE AT (0,0).

Suppose $\vec{C}_2$ = ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ traversed $\circlearrowleft$



Let $E = \{(x,y) : 1 \le x^2+y^2,\ \frac{x^2}{4}+\frac{y^2}{9} \le 1\}$

$M\,dx + N\,dy$ well-defined on E.

$$\int_{\vec{C}_2} M\,dx+N\,dy - \int_{\vec{C}_1} M\,dx+N\,dy = \iint_E \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \iint_E 0\,dx\,dy = 0,$$

so $\int_{\vec{C}_2} M\,dx+N\,dy = \int_{\vec{C}_1} M\,dx+N\,dy = 2\pi.$

DIVERGENCE THEOREM. Suppose that $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is smooth over a region D in (x,y) bounded by a piecewise smooth curve $\vec{C}$. If $\vec{N}$ is the outward-pointing unit normal to $\vec{C}$, then

$$\int_{\vec{C}} \vec{F}\cdot\vec{N}\,ds = \iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy = \iint_D \nabla\cdot\vec{F}\,dx\,dy$$

Proof: Let $*\vec{N}$ = $\vec{N}$ rotated counterclockwise thru $90°$

$*\vec{N} = \vec{T}$, where $\vec{T}$ is tangent to $\vec{C}$ with D lying on left.

$*\vec{F}(x,y) = -Q(x,y)\,\vec{i} + P(x,y)\,\vec{j} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$

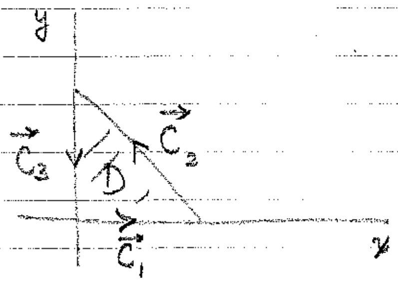
$$\int_{\vec{C}} \vec{F}\cdot\vec{N}\,ds = \int_{\vec{C}} (*\vec{F})\cdot\vec{T}\,ds = \iint_D \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy.$$

Suppose that $\vec{F} = (y \cos e^y)\,\vec{i} + (x + y^2)\,\vec{j}$.

What is $\int_C \vec{F}\cdot\vec{N}\,ds$, where $\vec{C} = \partial D$,

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x,\ 0 \le y,\ x+y \le 1\}$?



$$\partial D = \vec{C}_1 + \vec{C}_2 + \vec{C}_3$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y \cos e^y) + \frac{\partial}{\partial y}(x + y^2) = 0 + 2y$$

$$\int_{\vec{C}} \vec{F}\cdot\vec{N}\,ds = \iint_D 2y\,dx\,dy = \int_0^1 \left[\int_0^{1-x} 2y\,dy \right] dx$$

$$= \int_0^1 y^2 \Big|_0^{1-x} dx = \int_0^1 (1-x)^2\,dx = -\tfrac{1}{3}(1-x)^3 \Big|_0^1 = \boxed{\tfrac{1}{3}}$$

DIVERGENCE THEOREM IN $\mathbb{R}^3$. Let D be a region in $\mathbb{R}^3$ bounded by the piecewise smooth surface S, $\vec{N}$ the outward pointing unit normal S. If $\vec{F}(x,y,z)$ is smooth on D and its boundary,

$$\iint \vec{F}\cdot\vec{N}\,dA = \iiint_D \nabla\cdot\vec{F}\,dx\,dy\,dz$$

EXAMPLE: Find $\iint_S \vec{F}\cdot\vec{N}\,dA$, where S is the sphere $x^2+y^2+z^2 = 1$, $\vec{N}$ = outward normal, and

$$\vec{F}(x,y,z) = \begin{pmatrix} \sin y \\ e^{-x^2} \\ z^3 \end{pmatrix}.$$

SOLUTION. $S = \partial D$ where $D = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(\sin y) + \frac{\partial}{\partial y}(e^{-x^2}) + \frac{\partial}{\partial z}(z^3) = 3z^2$$

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \iiint_D 3z^2\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\left[\int_0^{\pi}\left[\int_0^1 3\rho^2\cos^2\phi\ \rho^2\sin\phi\,d\rho\right]d\phi\right]d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} \frac{3}{5}\rho^5\Big|_0^1 \cos^2\phi\sin\phi\,d\phi\ \right]d\theta$$

$$= \int_0^{2\pi} -\frac{1}{5}\cos^3\phi\Big|_0^{\pi} d\theta = \int_0^{2\pi}\frac{2}{5}\,d\theta = \boxed{\frac{4}{5}\pi}.$$

DIVERGENCE THM GIVES AN INTERPRETATION OF $\nabla\cdot\vec{F}$

Suppose $\vec{V}(x,y,z)$ = velocity of fluid $\rho(x,y,z)$ = density of fluid.

Then if $\vec{F} = \rho\vec{V}$,

$\iint_S \vec{F}\cdot\vec{N}\,dA$ = rate of flow of flow of fluid across S in direction of $\vec{N}$.

Suppose $(x_0,y_0,z_0) \in \mathbb{R}^3$

$D_\varepsilon = \{(x,y,z) : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \le \varepsilon\}$

$S_\varepsilon = \partial S_\varepsilon$.

$$\nabla\cdot\vec{F}(x_0,y_0,z_0) = \lim_{\varepsilon\to 0}\frac{\iiint_{D_\varepsilon}\nabla\cdot\vec{F}\,dx\,dy\,dz}{\text{Vol of } D_\varepsilon} = \lim_{\varepsilon\to 0}\frac{\iint_{S_\varepsilon}\vec{F}\cdot\vec{N}\,dA}{\text{Vol of } D_\varepsilon}$$

= rate of creation of fluids per unit volume at (x_0,y_0,z_0).