Math. 5B Friday, May 20, 2011

$d\vec{x} = d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\uparrow$ Lovric notation

In $\mathbb{R}^2$, $d\vec{x} = d\vec{s} = dx\,\vec{i} + dy\,\vec{j}$

If $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$

$\vec{F}\cdot d\vec{x} = \vec{F}\cdot d\vec{r} = M(x,y)\,dx + N(x,y)\,dy.$

$\uparrow$ This is a <u>differential</u>

It is an integrand for a line integral.

<u>Green's Theorem.</u> Suppose D is a region in the (x,y)-plane bounded by a piecewise smooth curve $\vec{C} = \partial\vec{D}$ directed so that as $\vec{C}$ is traversed in positive direction D is on the left. Then

$$\int_{\vec{C}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Example: Suppose $\vec{C}$ is the counterclockwise unit circle $x^2+y^2=1$

What is $\int_{\vec{C}} (-y\,dx + x\,dy)$?

Method I. $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $0 \le t \le 2\pi$ is a counterclockwise parametrization.

$$\begin{cases} dx = -\sin t\, dt \\ dy = \cos t\, dt \end{cases}$$

$$-y\,dx + x\,dy = -\sin t(-\sin t\, dt) + \cos t(\cos t\, dt) = \cdots = dt.$$

$$\int_{\vec{C}} (-y\,dx + x\,dy) = \int_0^{2\pi} dt = 2\pi$$

Method II. $\vec{C} = \partial \vec{D}$, where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

$$\int_{\vec{C}} (-y\,dx + x\,dy) = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right] dx\,dy$$

$M = -y$

$N = x$

$$= \iint_D 2\,dx\,dy = 2 \text{ Area of } D = 2\pi.$$

Example 2: If $\vec{C}$ is a unit circle, what is

$$\int_{\vec{C}} (e^{-x^2} dx + x\,dy).$$

Method I is impossible

Method II: $M = e^{-x^2}$, $N = x$

$$\int_{\vec{C}} (e^{-x^2} dx + x\,dy) = \int_{\vec{C}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(e^{-x^2}) \right] dx\,dy = \iint_D dx\,dy$$

$$= \text{Area of } D = \pi.$$

Can use Green's theorem to calculate the area of a region D bounded by a piecewise smooth curve $\vec{C} = \overrightarrow{\partial D}$ traversed ↺

If $M = -\frac{1}{2}y$ and $N = \frac{1}{2}x$, then

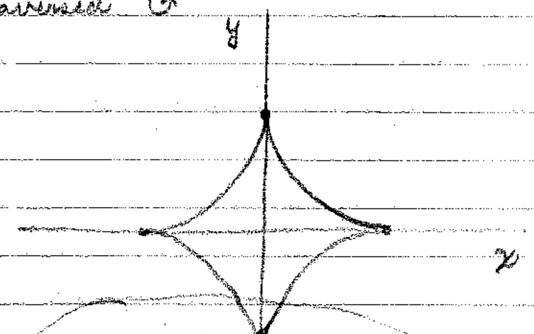
$$\int_{\overrightarrow{\partial D}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(\tfrac{1}{2}x\right) - \frac{\partial}{\partial y}\left(-\tfrac{1}{2}y\right)\right] dx\,dy$$

$$= \iint_D 1\,dx\,dy$$

Example: Find area of $D = \{(x,y) \in \mathbb{R}^2 : \sqrt[3]{x^2} + \sqrt[3]{y^2} \le 1\}$

$\vec{C} = \overrightarrow{\partial D}$ is the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ traversed ↺



Parametrization of $\vec{C}$

$x = \cos^3 t$
$y = \sin^3 t \qquad 0 \le t \le 2\pi$

$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$
$\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$

$dx = -3\cos^2 t \sin t\,dt$
$dy = 3\sin^2 t \cos t\,dt$

$-y\,dx + x\,dy = -3\sin^4 t \cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt$

$\sin 2t = 2\sin t\cos t$

$= 3\sin^2 t\cos^2 t\,dt = \frac{3}{4}(\sin 2t)^2\,dt$

$= \frac{3}{8}(1 - \cos 4t)\,dt$

Area of D $= \int_0^{2\pi} \frac{3}{16}(1-\cos 4t)\,dt = \cdots = \boxed{\frac{3}{8}\pi}$