Math. 5B — Wednesday, May 18, 2011

ORIENTED SURFACE INTEGRALS

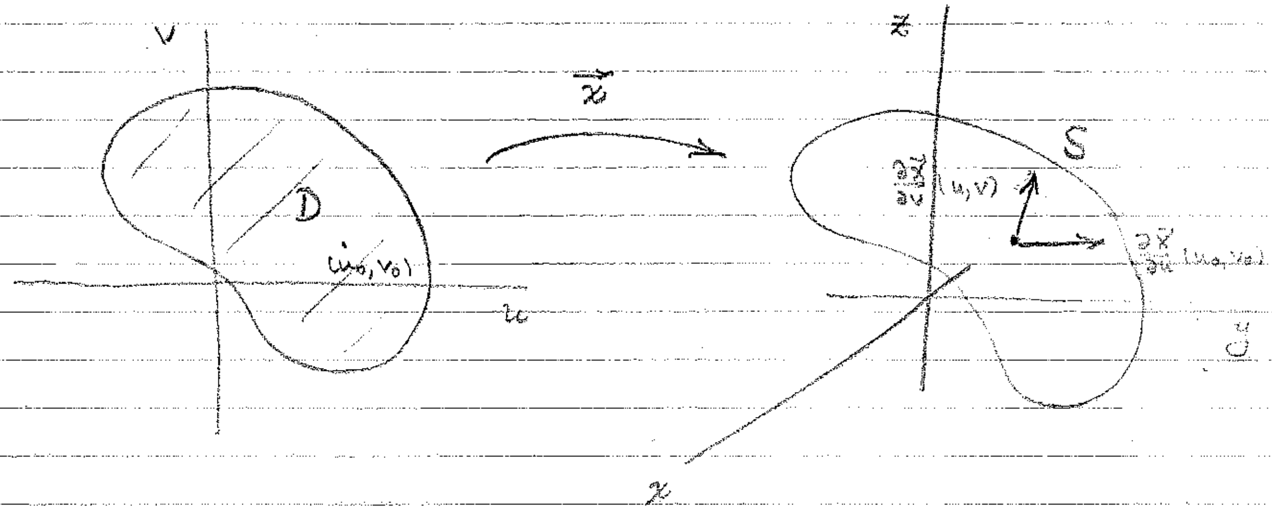
INTEGRALS OF VECTOR FIELDS OVER SURFACES.

READ LOVRIC, § 7.4.

If $\vec{v}$ & $\vec{w} \in \mathbb{R}^3$, then $\vec{v} \times \vec{w}$ is $\perp$ to $\vec{v}$ and $\vec{w}$ and

$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$, where $\theta = \angle$ between $\vec{v}$ and $\vec{w}$

$=$ Area of $\square$ spanned by $\vec{v}$ and $\vec{w}$.



$\vec{x} : D \longrightarrow S \subseteq \mathbb{R}^3$ a parametrization of S.

$\dfrac{\partial \vec{x}}{\partial u}(u_0, v_0)$ and $\dfrac{\partial \vec{x}}{\partial v}(u_0, v_0)$ are tangent to S at $\vec{x}(u_0, v_0)$

So $\dfrac{\partial \vec{x}}{\partial u} \times \dfrac{\partial \vec{x}}{\partial v}(u_0, v_0)$ is $\perp$ to S at $\vec{x}(u_0, v_0)$.

$$\vec{N}(u_0, v_0) = \frac{\dfrac{\partial \vec{x}}{\partial u} \times \dfrac{\partial \vec{x}}{\partial v}}{\left|\dfrac{\partial \vec{x}}{\partial u} \times \dfrac{\partial \vec{x}}{\partial v}\right|}(u_0, v_0) = \text{unit normal to S at } \vec{x}(u_0, v_0).$$

Two possible choices of unit normal, $\vec{N}$ or $-\vec{N}$.

Choice of unit normal gives an orientation to the surface.

Suppose that $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a vector field on $\mathbb{R}^3$.

The <u>flux</u> of $\vec{F}$ through S in the direction of $\vec{N}$ is the surface integral

$$\iint_S \vec{F} \cdot \vec{N}\, dA$$

But $dA = \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| du\,dv \qquad \vec{N} = \dfrac{\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}}{\left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right|}$,

so $\vec{N}\, dA = \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\, du\,dv$, and hence

$$\iint_S \vec{F} \cdot \vec{N}\, dA = \iint_D F \cdot \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\, du\,dv.$$

Example: S = sphere $x^2+y^2+z^2 = 1$

$\vec{N} = y\vec{i} - x\vec{j} + z\vec{k}$ What is $\iint_S \vec{F} \cdot \vec{N}\, dA$?

Need to parametrize S. Use spherical coordinates

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned} \qquad \vec{x}(\phi,\theta) = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix}.$$

$$x^2 + y^2 + z^2 = 1 \Rightarrow |\vec{x}(\phi, \theta)| = 1.$$

$$\frac{\partial \vec{x}}{\partial \phi} = \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ -\sin\phi \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial \theta} = \begin{pmatrix} -\sin\phi \cos\theta \\ \sin\phi \sin\theta \\ 0 \end{pmatrix} = \sin\phi \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \theta} = \sin\phi \begin{vmatrix} \vec{i} & \cos\phi \cos\theta & -\sin\theta \\ \vec{j} & \cos\phi \sin\theta & \cos\theta \\ \vec{k} & -\sin\phi & 0 \end{vmatrix} = \sin\phi \left(\sin\phi \cos\theta \, \vec{i} + \sin\phi \sin\theta \, \vec{j} + \cos\phi \, \vec{k} \right)$$

$$= \sin\phi \, \vec{x}(\phi, \theta)$$

$$\vec{N} \, dA = \sin\phi \, \vec{x}(\phi, \theta) \, d\phi \, d\theta.$$

$$\vec{F} \cdot \vec{N} \, dA = \sin\phi \begin{pmatrix} y \\ -x \\ z \end{pmatrix} \cdot \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} d\phi \, d\theta$$

$$= \sin\phi \begin{pmatrix} \sin\phi \sin\theta \\ -\sin\phi \cos\theta \\ \cos\phi \end{pmatrix} \cdot \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} d\phi \, dA$$

$$= \sin\phi \cos^2\phi \, dA$$

$$\iint_S \vec{F} \cdot \vec{N} \, dA = \int_0^{2\pi} \left[\int_0^{\pi} \sin\phi \cos^2\phi \, d\phi \right] d\theta = \int_0^{2\pi} -\frac{1}{3} \cos^3\phi \Big|_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} \, d\theta = \boxed{\frac{4}{3}\pi}.$$

Interpretation of $\iint_S \vec{F} \cdot \vec{N} \, dA$

Suppose $\vec{V}(x, y, z)$ is the velocity of a steady-state fluid

$\rho(x, y, z)$ = density of fluid

(density) (normal component of velocity) $dA = \rho \, \vec{V} \cdot \vec{N} \, dA$

If $\vec{F} = \rho\vec{V}$, $\iint_S F \cdot \vec{N}\, dA$ = rate of fluid flow across S in direction of $\vec{N}$.

PROBLEMS FOR FRIDAY'S QUIZ.

1. If $\vec{V} = (1,1,0)$ and $\vec{W} = (1,0,1)$ what is area of $\square$ spanned by $\vec{V}$ & $\vec{W}$?

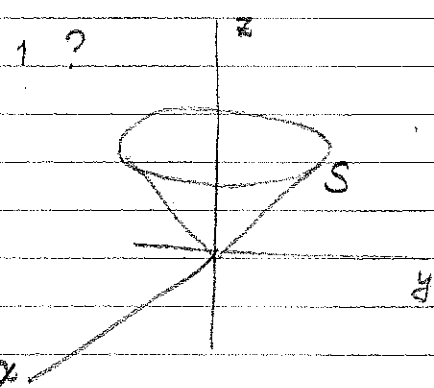
Answer: $|\vec{V} \times \vec{W}| = \cdots = \sqrt{3}$

2. If S = cone (or some other surface) what is $\iint_S z\, dA$?

3. If $\vec{F}$ is a given vector field, S a given surface, what is $\iint_S \vec{F} \cdot \vec{N}\, dA$

Hard part in 2 & 3 might be finding parametrization

Suppose S is $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$?



$$\vec{x}(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u \end{pmatrix} \qquad \begin{array}{l} 0 \le u \le 1 \\ 0 \le v < 2\pi \end{array}$$

Can you find parametrizations like this?