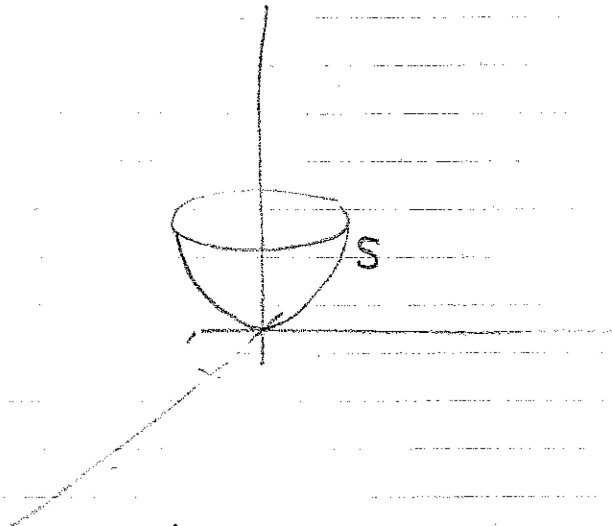
Math. 5B

Monday, May 16, 2011



PARABOLOID OF REVOLUTION

$z = x^2 + y^2 \ , \quad x^2 + y^2 < 1$

SIMPLEST PARAMETRIZATION

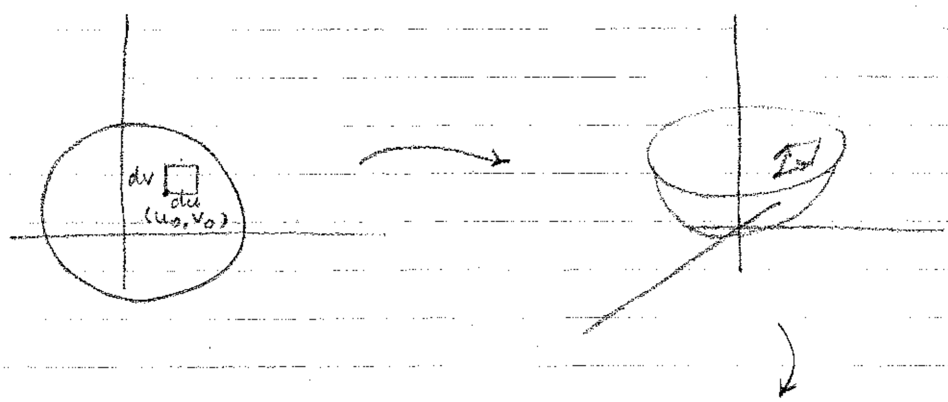
$D = \{ (u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1 \}$

$x(u,v) = u$
$y(u,v) = v$
$z(u,v) = u^2 + v^2$

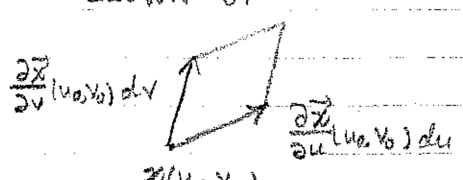
$$\vec{x}(u,v) = \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix}$$

$\vec{x} : D \longrightarrow \mathbb{R}^3.$

SUPPOSE WE WANT TO FIND AREA OF PARABOLOID



BLOWN UP



Area of $\square$ spanned by

$\frac{\partial \vec{x}}{\partial u}(u_0,v_0)\,du$ and $\frac{\partial \vec{x}}{\partial v}(u_0,v_0)\,dv$

is $\left| \frac{\partial \vec{x}}{\partial u}(u_0,v_0) \times \frac{\partial \vec{x}}{\partial v}(u_0,v_0) \right| du\,dv$

$$dA = \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| du\,dv$$

If $\vec{x}(u,v) = \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix}$, $\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}$ $\frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix}$

$$\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} = \begin{vmatrix} \vec{i} & 1 & 0 \\ \vec{j} & 0 & 1 \\ k & 2u & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$$\left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| = \sqrt{1+4u^2+4v^2}$$

Area of PARABOLOID $S = \iint_D \sqrt{1+4u^2+4v^2}\, du\,dv$

$$= \int_0^{2\pi} \left[\int_0^1 \sqrt{1+4r^2}\; r\,dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 \frac{1}{8} (1+4r^2)^{\frac{1}{2}}\, 8r\,dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8} \cdot \frac{2}{3} (1+4r^2)^{\frac{3}{2}} \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (5^{3/2} - 1)\, d\theta = \boxed{\frac{\pi}{6} (5\sqrt{5} - 1)}$$

THERE ARE TWO TYPES OF INTEGRALS OVER A SURFACE $S \subseteq \mathbb{R}^3$

I. UNORIENTED INTEGRALS

If $f(x,y,z)$ is a function on $\mathbb{R}^3$

$$\iint_S f(x,y,z)\, dA = \iint_D f(x(u,v), y(u,v), z(u,v)) \left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| du\,dv$$

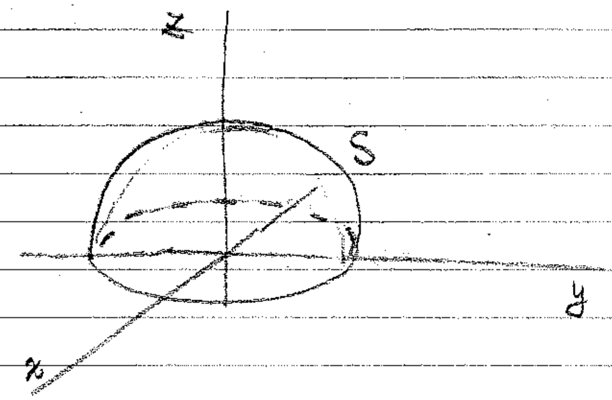
when $\vec{x} : D \longrightarrow S$ is a parametrization of S with $\vec{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$.

II. ORIENTED INTEGRALS (TREATED NEXT TIME)

Suppose S is the upper hemisphere : $x^2+y^2+z^2=1$, $z \geq 0$

What is

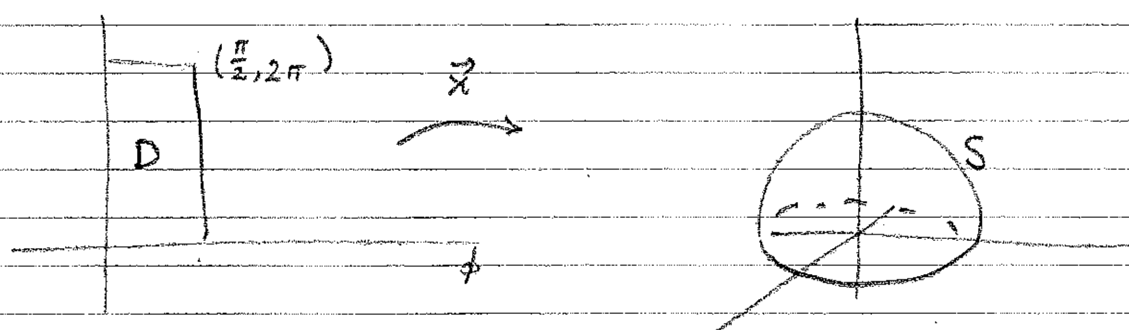
$$\iint_S z \, dA \ ?$$



To answer this question, we need a parametrization of S :

SPHERICAL COORDS : $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$ $\quad S : \rho = 1$

$$\begin{cases} x = \sin\phi \cos\theta \\ y = \sin\phi \sin\theta \\ z = \cos\phi \end{cases} \qquad \vec{x}(\phi,\theta) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \qquad \begin{matrix} 0 < \phi < \frac{\pi}{2} \\ 0 < \theta < \pi \end{matrix}$$



$$\frac{\partial \vec{x}}{\partial \phi} = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial \theta} = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \theta} = \begin{vmatrix} \vec{i} & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \vec{j} & \cos\phi\sin\theta & \sin\phi\cos\theta \\ \vec{k} & -\sin\phi & 0 \end{vmatrix} = \sin\phi \begin{vmatrix} \vec{i} & \cos\phi\cos\theta & -\sin\theta \\ \vec{j} & \cos\phi\sin\theta & \cos\theta \\ k & -\sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi \left(\sin\phi\cos\theta \, \vec{i} + \sin\phi\sin\theta \, \vec{j} + \cos\phi \, \vec{k} \right)$$

$$\left| \frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \theta} \right| = \sin\phi$$

$$\iint_S z\,dA = \iint_D \cos\phi\,(\sin\phi\,d\phi\,d\theta)$$

$$= \int_0^{2\pi}\left[\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right]d\theta$$

$$= \int_0^{2\pi}\left[-\tfrac{1}{2}\cos^2\phi\right]_0^{\pi/2}d\theta = \int_0^{2\pi}\tfrac{1}{2}\,d\theta = \pi.$$

$$\bar{z} = z\text{-component of center of mass} = \frac{\iint_S z\,dA}{\iint_S dA} = \frac{\pi}{2\pi} = \boxed{\frac{1}{2}}$$