Math. 5B Wednesday, March 30, 2011

If $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ $\quad \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

$\|\vec{v} - \vec{w}\| =$ Euclidean distance from $\vec{v}$ to $\vec{w}$

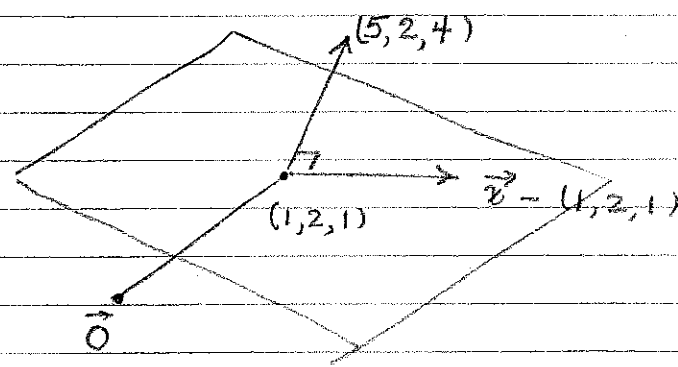
$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$, where $\theta = \angle$ between $\vec{v}$ & $\vec{w}$.

Suppose $\vec{v} \neq \vec{0}$ and $\vec{w} \neq 0$.

Then $\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$

Suppose $\vec{x} = (x, y, z)$

When is $\vec{x} - (1,2,1) \perp$ to $(5,2,4)$?




$(5, 2, 4) \cdot [\vec{x} - (1, 2, 1)] = 0$

$(5, 2, 4) \cdot [(x, y, z) - (1, 2, 1)] = 0.$

$(5, 2, 4) \cdot (x-1, y-2, z-1) = 0$

$5(x-1) + 2(y-2) + 4(z-1) = 0$

$5x + 2y + 4z - 5 - 4 - 4 = 0$

$\boxed{5x + 2y + 4z = 13}$

Equations of plane which passes thru $(1, 2, 1)$ and is $\perp$ to $(5, 2, 4)$ is $5x + 2y + 4z = 13$.

Standard orthonormal basis for $\mathbb{R}^3$

$\vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1)$

$(3, 7, 4) = 3\vec{i} + 7\vec{j} + 4\vec{k}$.

If $\vec{v} = (v_1, v_2, v_3), \quad \vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} + \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_3 \end{vmatrix} \vec{k}$$

$\vec{v} = (1, 0, 5), \quad \vec{w} = (2, 1, 3)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 5 \\ 2 & 1 & 3 \end{vmatrix} = -5\vec{i} + 7\vec{j} + \vec{k}$$

Properties of cross product:

1. $\vec{v} \times \vec{w}$ is $\perp$ to $\vec{v}$ and to $\vec{w}$.

2. $\| \vec{v} \times \vec{w} \| = \| \vec{v} \| \, \| \vec{w} \| \, |\sin \theta|$

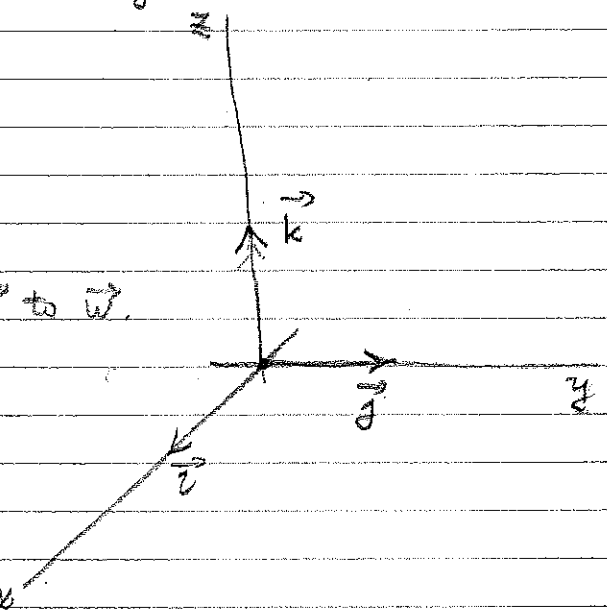
where $\theta$ is $\angle$ between $\vec{v}$ and $\vec{w}$

1 & 2 determine $\vec{v} \times \vec{w}$ up to sign.

3. RIGHT HAND RULE

points in dir of thumb

when fingers curl from $\vec{v}$ to $\vec{w}$.




Find a vector $\perp$ to $\vec{v} = (1, 0, 6)$ and $\vec{w} = (0, 1, 4)$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 6 \\ 0 & 1 & 4 \end{vmatrix} = -6\vec{i} - 4\vec{j} + \vec{k}.$$

IMPORTANT FACT

$$\| \vec{v} \times \vec{w} \| = \| \vec{v} \| \, \| \vec{w} \| \, |\sin \theta|$$

$= $ Area of $\square$ spanned by $\vec{v}$ and $\vec{w}$.

Suppose $f$ is a real-valued function of two variables.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Example: $f(x,y) = x^2 e^y$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{(x+h)^2 e^y - x^2 e^y}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2)e^y - x^2 e^y}{h}$$

$$= \lim_{h\to 0} \frac{2xhe^y + h^2 e^y}{h} = \lim_{h\to 0} 2xe^y + he^y$$

$$= 2xe^y$$

Rule of thumb: differentiate w.r.t. $x$ treating $y$ as constant.

$$\frac{\partial f}{\partial y}(x,y) = \cdots = x^2 e^y.$$

Partial derivatives can be used to find tangent planes to surfaces.

We want to study the geometry of a surface S :

$z = f(x, y)$

Example : $z = \frac{1}{4}(x^2+y^2) = f(x,y)$

Level sets : $f(x,y) = c.$

$\frac{1}{4}(x^2+y^2) = c$

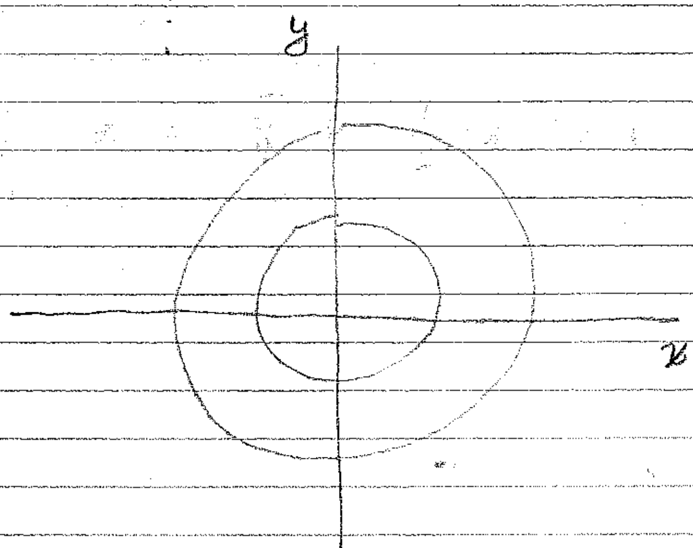
$x^2+y^2 = 4c.$

$c = 0 \quad : \quad x^2+y^2 = 0$

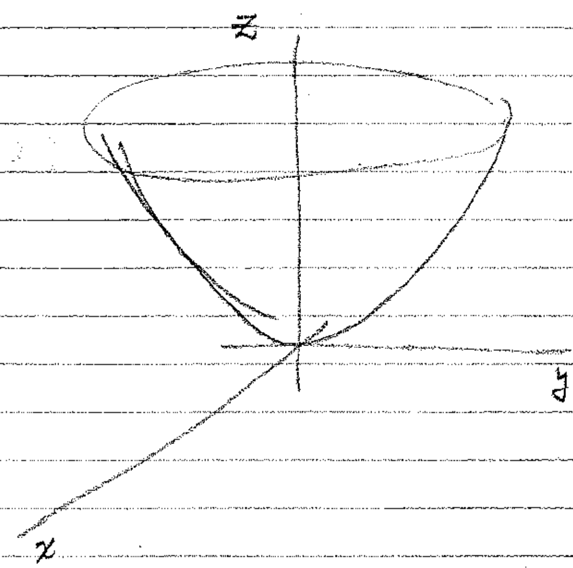
$c = \frac{1}{4} \quad : \quad x^2+y^2 = 1$

$c = 1 \quad : \quad x^2+y^2 = 4$

$\vdots$




topographic map of f




paraboloid of revolution.

Theorem. Suppose a surface S is represented by

$$z = f(x, y)$$

Then the equation of tangent plane to S at $(x_0, y_0, f(x_0, y_0))$ is

$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}$$

Example: $z = \frac{1}{4}(x^2 + y^2) = f(x,y)$

Tangent plane at $(1, 2, \frac{5}{4})$?

$\frac{\partial f}{\partial x} = \frac{1}{2} x$

$\frac{\partial f}{\partial y} = \frac{1}{2} y$

$$\boxed{z = \frac{5}{4} + \frac{1}{2}(x - 1) + (y - 2).}$$