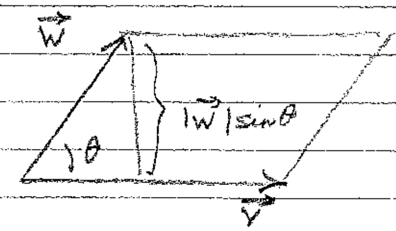
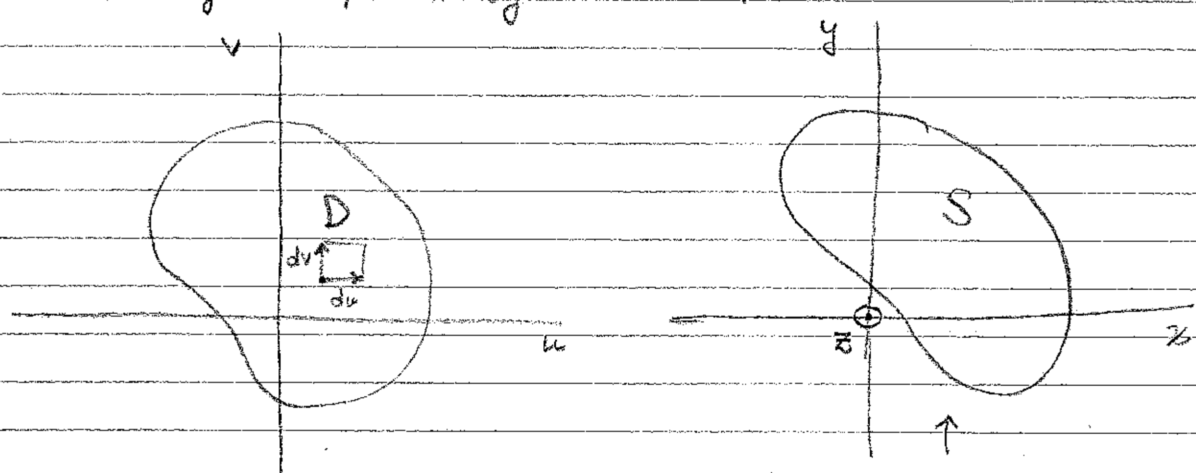
Math. 5B — Monday, June 9, 2011

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$



$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$$

= area of ▱ spanned by $\vec{v}$ and $\vec{w}$.



Think of S as a surface $\subseteq$ (x,y) – plane

Suppose

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

give a 1-1 correspondence between D and S

$$\vec{x} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ 0 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ 0 \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ 0 \end{pmatrix}$$

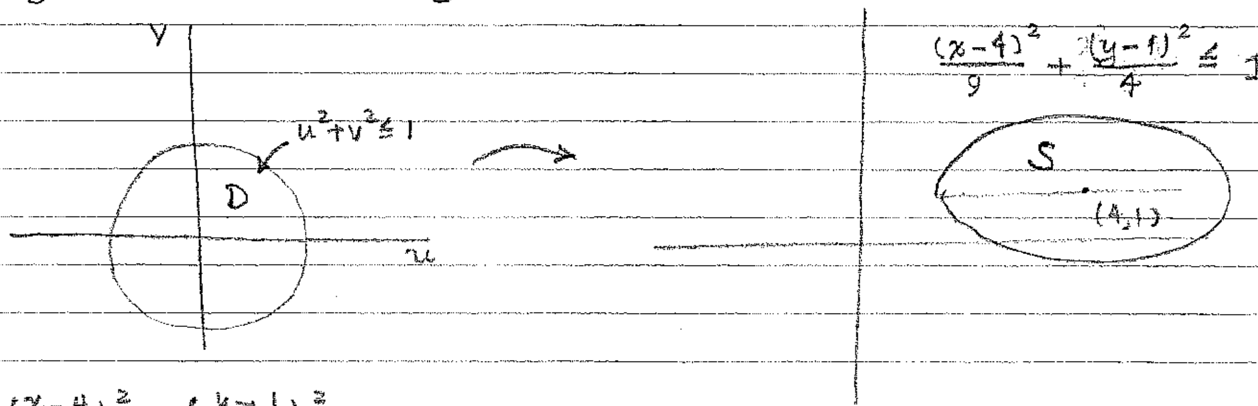
$$\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} = \begin{vmatrix} \vec{i} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \vec{j} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \vec{k} & 0 & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k} = \frac{\partial(x,y)}{\partial(u,v)} \vec{k}$$

Area of ▱ spanned by $\frac{\partial \vec{x}}{\partial u} du$ and $\frac{\partial \vec{x}}{\partial v} dv$

is $\left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| du\, dv = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$

Area element $dx\, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

$$\iint_S f(x,y)\,dx\,dy = \iint_D f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



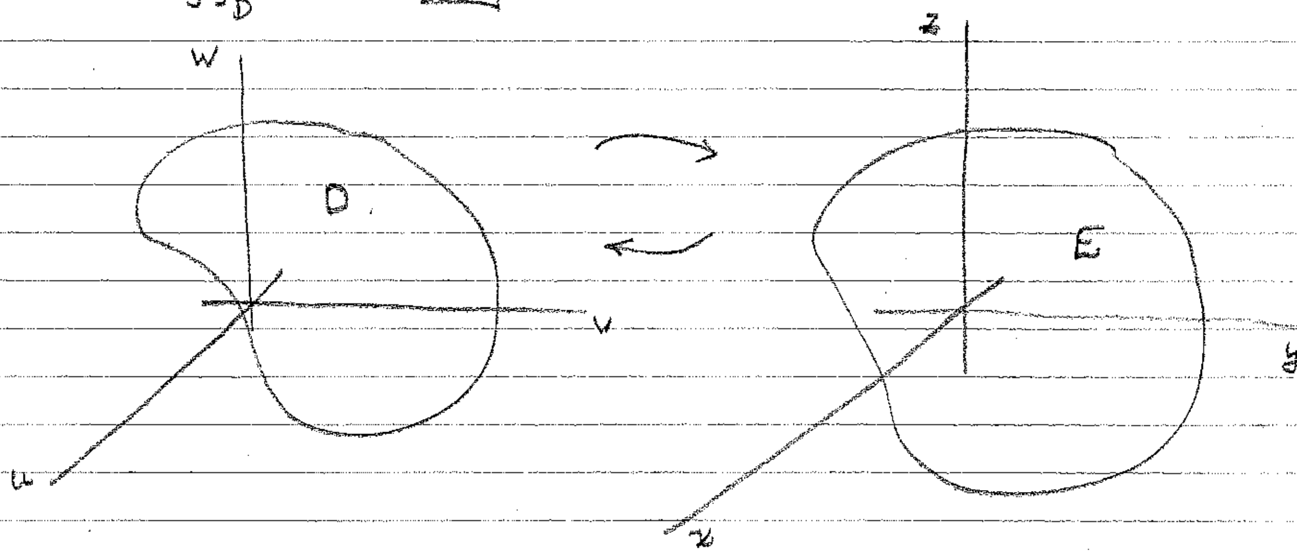
$$\left(\frac{x-4}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2 \le 1$$

$$u^2 + v^2 \le 1$$

$$\begin{cases} u = \dfrac{x-4}{3} \\ v = \dfrac{y-1}{2} \end{cases} \qquad \begin{cases} x = 3u+4 \\ y = 2v+1 \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

Area bounded by ellipse $= \iint_S 1\,dx\,dy = \cdots = \iint_D 6\,du\,dv$

$$= 6\iint_D du\,dv = \boxed{6\pi}$$



$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad \begin{cases} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{cases}$$

give 1 − 1 correspondence between D and E

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_D f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Find volume enclosed by ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a, b, c > 0$$

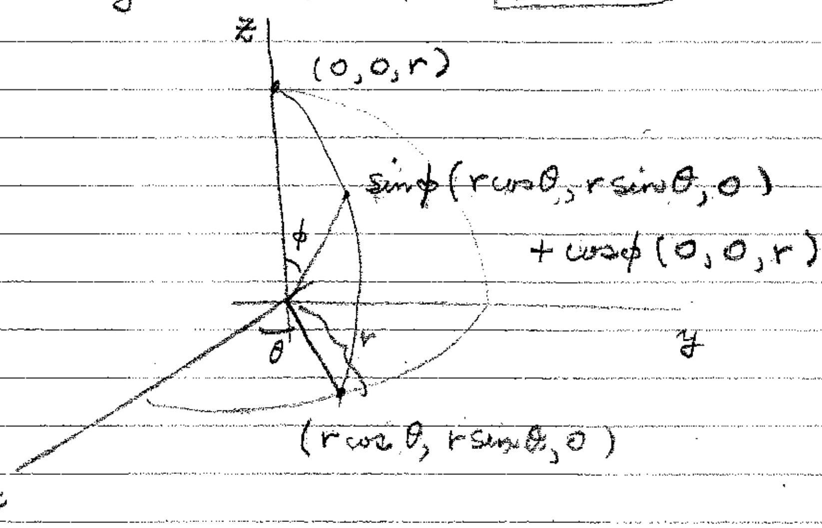
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \qquad u^2 + v^2 + w^2 = 1$$

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \\ w = \frac{z}{c} \end{cases} \qquad \begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\text{Volume enclosed by ellipsoid} = \iiint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2<1} dx\,dy\,dz = \iiint_{u^2+v^2+w^2<1} abc\,du\,dv\,dw$$

$$= abc\,(\text{volume enclosed by unit sphere}) = \boxed{\frac{4}{3}\pi abc}$$

$$\begin{cases} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$



What is $\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}$?

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \cdots = r^2\sin\phi. \qquad dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$(\text{Volume inside } x^2+y^2+z^2=a^2) = \iiint_{x^2+y^2+z^2\le 1} dx\,dy\,dz$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^{a} r^2 \sin\phi \, dr \right] d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3} r^3 \sin\phi \Big|_0^{a} \, d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3} a^3 \sin\phi \, d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} a^3 \cos\phi \Big|_0^{\pi} \right] d\theta = \int_0^{2\pi} \frac{2}{3} a^3 \, d\theta = \boxed{\frac{4}{3} \pi a^3}$$