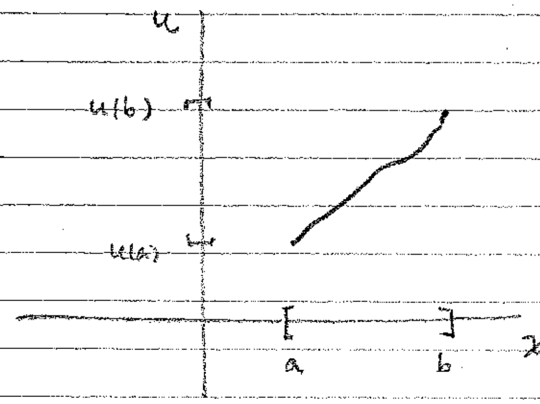
Math. 5B — Friday, May 6, 2011



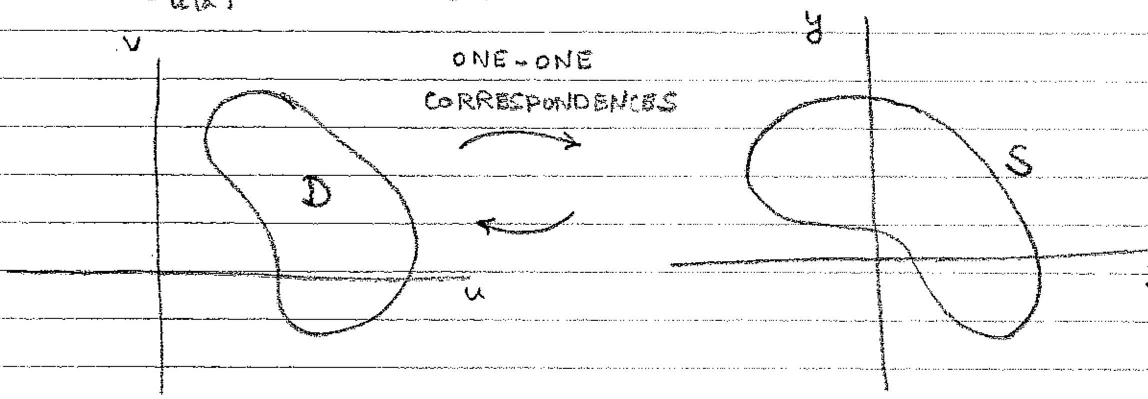
Suppose $x = x(u)$, $u = u(x)$

give one-one correspondence between

$a \le x \le b$ and $u(a) \le u \le u(b)$

Then

$$\int_a^b f(x)\,dx = \int_{u(a)}^{u(b)} f(x(u))\,\frac{dx}{du}\,du$$



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

Then

$$\iint_S f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

If $dA = dx\,dy$ in (x,y)-plane $\quad dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$

EXAMPLE: Suppose we want to find area of region S

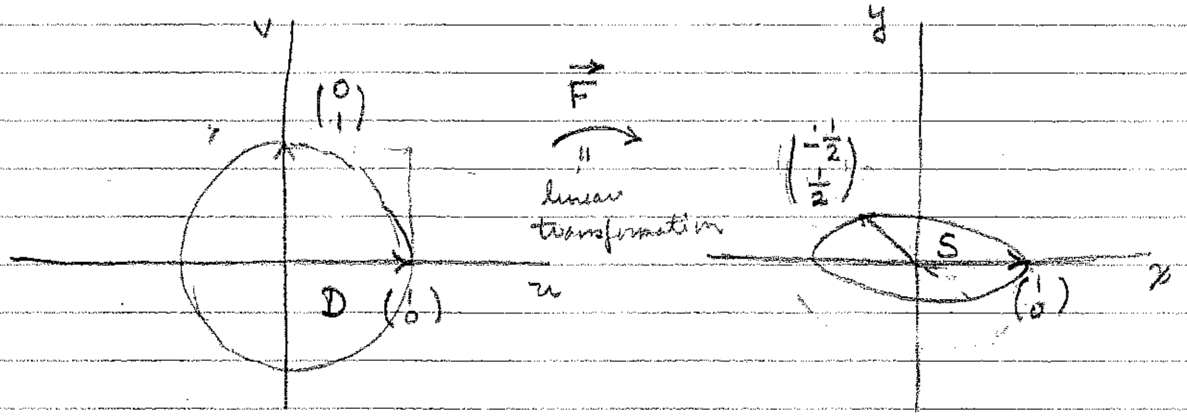
bounded by $x^2 + 2xy + 5y^2 = 1$

$(x^2 + 2xy + y^2) + 4y^2 = 1$

$(x+y)^2 + (2y)^2 = 1$ $\qquad\qquad u^2 + v^2 = 1$

$$\begin{cases} u = x+y \\ v = 2y \end{cases} \qquad \begin{cases} x = u - \frac{1}{2}v \\ y = \frac{1}{2}v \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$



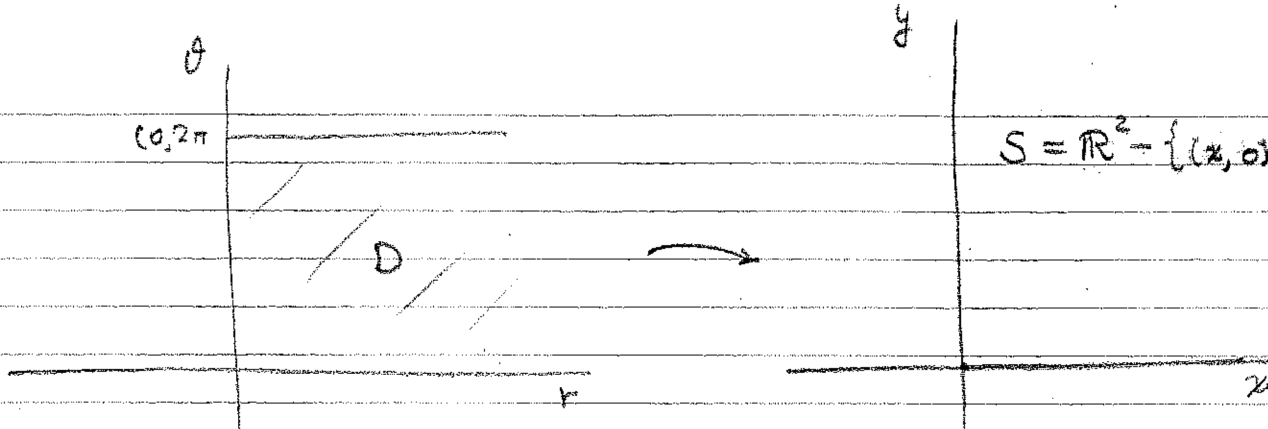
$D = \{(u,v) : u^2 + v^2 \le 1\}$. $\qquad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \det(DF) = \begin{vmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

$$\text{Area of } S = \iint_S 1 \, dx\,dy = \iint_D \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_D \frac{1}{2} \, du\,dv = \frac{1}{2} \iint_D du\,dv = \frac{1}{2}\pi$$

Here we have used the fact that the area of the unit disk is π.

Suppose we wanted to find this out directly.



$S = \mathbb{R}^2 - \{(x,0) : x \geq 0\}$

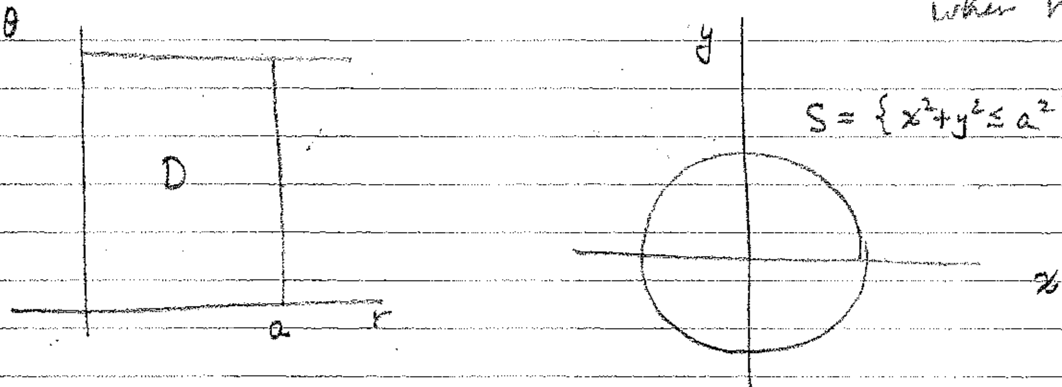
$D = \{(r,\theta) : r > 0,\ 0 < \theta < 2\pi\}$

$$\vec{F}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$D\vec{F} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left| \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| = \left| r(\cos^2\theta + \sin^2\theta) \right| = r$$

when $r > 0$



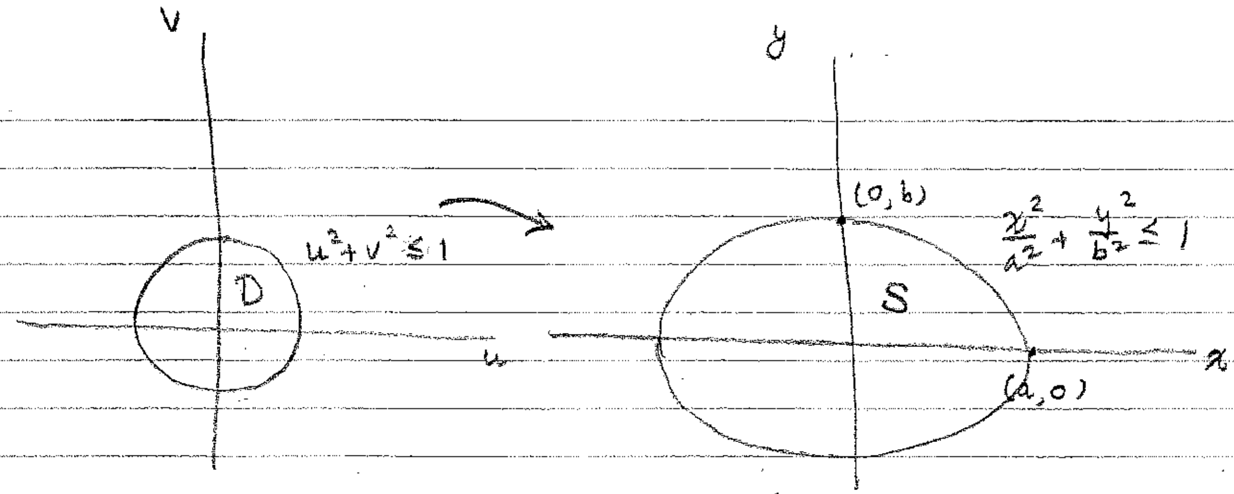
$S = \{x^2+y^2 \leq a^2\}$

$D = \{(r,\theta) : 0 < r \leq a,\ 0 < \theta < 2\pi\}$

$$\text{Area of } S = \iint_S 1\,dx\,dy = \iint_D \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_D r\,dr\,d\theta = \int_0^{2\pi}\int_0^a r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big]_0^a d\theta = \int_0^{2\pi} \frac{1}{2}a^2\,d\theta = \pi a^2.$$



What is area of $S = \{(x,y) : \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1\}$

$a > 0,\ b > 0.$

Let $\frac{x}{a} = u \quad \frac{y}{b} = v$

$$\begin{cases} x = au \\ y = bv \end{cases} \qquad \begin{cases} u = \frac{1}{a}x \\ v = \frac{1}{b}y \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\text{Area of } S = \iint_S dx\,dy = \iint_D \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = ab \iint_D du\,dv$$

$$= \boxed{\pi ab}.$$