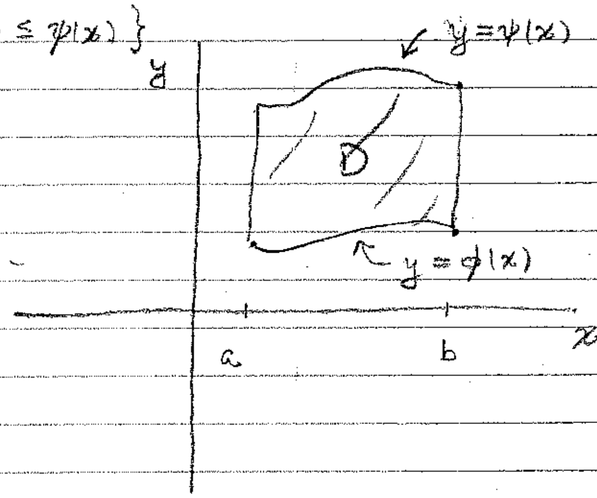
Math. 5B — Wednesday, May 4, 2011

D a bounded region in (x,y)-plane

Type I: $D = \{(x,y) : a \le x \le b, \ \phi(x) \le y \le \psi(x)\}$

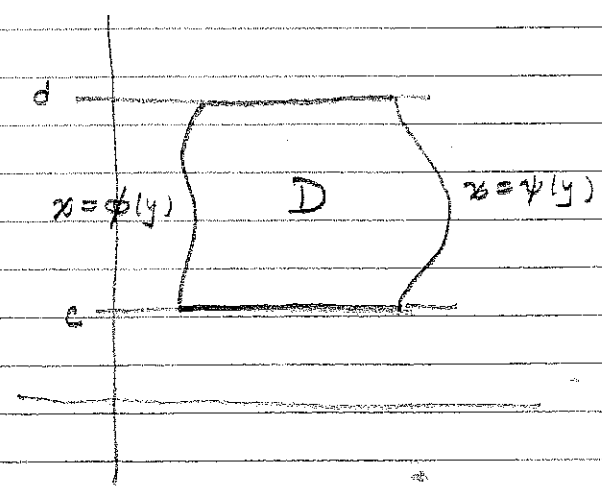


$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_D f(x,y)\,dA$$

$$= \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x,y)\,dy\right] dx$$

Type II: $D = \{(x,y) : c \le y \le d, \ \phi(y) \le x \le \psi(y)\}$



$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_D f(x,y)\,dA$$

$$= \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x,y)\,dx\right] dy$$

In general if D is a union of Type I and type II regions

$D = D_1 \cup \cdots \cup D_k$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \cdots + \iint_{D_k} f(x,y)\,dx\,dy.$$

If $f(x,y) = 1$, $\iint_D 1 \, dx\,dy$ = Area of D

If $f(x,y) \geq 0$, $\iint_D f(x,y)\,dx\,dy$ = Volume under $f(x,y)$ and over D.

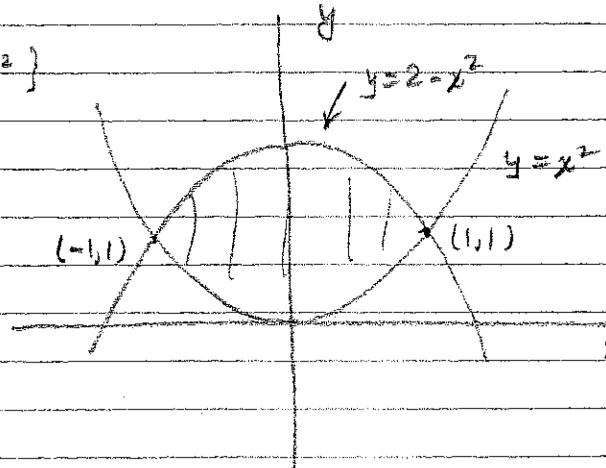
= Volume of $E = \{(x,y,z) : (x,y) \in D,\ z \leq f(x,y)\}$

If $f(x,y)$ = mass density at (x,y), $\iint_D f(x,y)\,dx\,dy$ = total mass

Average value of f on D = $\dfrac{\iint_D f(x,y)\,dx\,dy}{\iint_D 1\,dx\,dy}$

Example : $D = \{(x,y) : x^2 \leq y \leq 2 - x^2\}$

What is Area of D ?



Area of D

$$= \int_{-1}^{1} \left[\int_{x^2}^{2-x^2} dy \right] dx$$

$$= \int_{-1}^{1} [\,y\,]_{x^2}^{2-x^2}\, dx = \int_{-1}^{1} [2 - x^2 - x^2]\, dx$$

$$= \int_{-1}^{1} 2 - 2x^2\, dx = \left[2x - \tfrac{2}{3}x^3\right]_{-1}^{1} = \tfrac{4}{3} - \left(-\tfrac{4}{3}\right) = \boxed{\tfrac{8}{3}}.$$

Example: Find volume of

$E = \{(x,y,z) : (x,y) \in D$ and $0 \leq z \leq x^2\}$.

SOLUTION:

$$\text{Volume of } E = \iint_D x^2 \, dx\,dy$$

$$= \int_{-1}^{1} \left[\int_{x^2}^{2-x^2} x^2 \, dy \right] dx$$

$$= \int_{-1}^{1} \Big[x^2 y \Big]_{x^2}^{2-x^2} dx$$

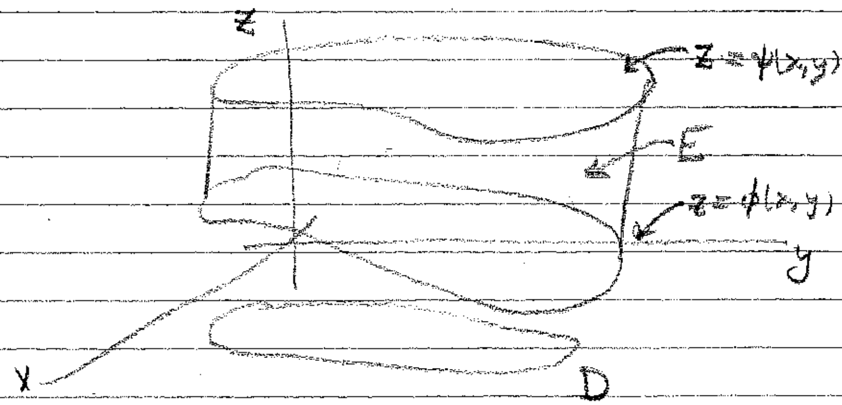
$$= \int_{-1}^{1} [x^2(2-x^2) - x^4] \, dx = \int_{-1}^{1} (2x^2 - 2x^4) \, dx$$

$$= \left[\frac{2}{3} x^3 - \frac{2}{5} x^5 \right]_{-1}^{1} = \frac{4}{3} - \frac{4}{5} = \boxed{\frac{8}{15}}$$

If E is a region in (x, y, z) - space.

We say E is of type I if

$$E = \{ (x,y,z) : (x,y) \in D, \ \phi(x,y) \le z \le \psi(x,y) \}$$

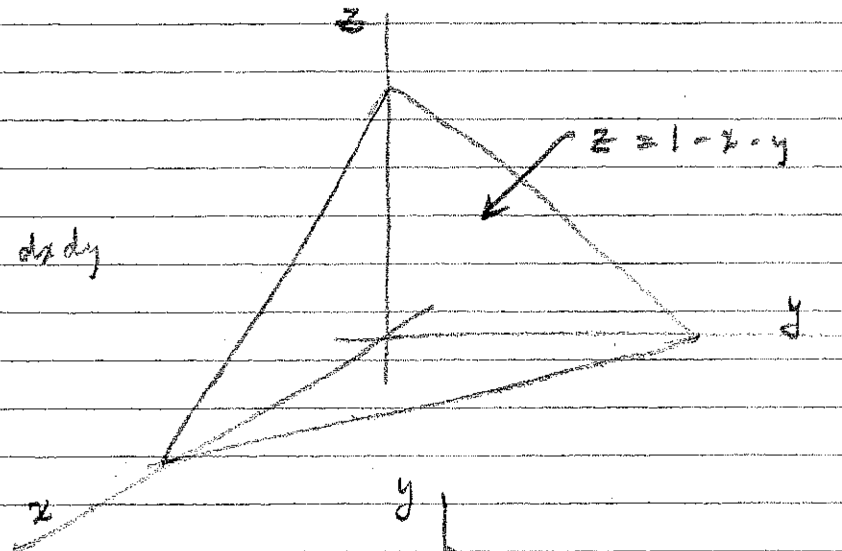


$$\iiint_E f(x,y,z) \, dx\,dy\,dz = \iiint_E f(x,y,z) \, dV$$

$$= \iint_D \left[\int_{\phi(x,y)}^{\psi(x,y)} dz \right] dx\,dy.$$

Similarly there are type II and type III regions

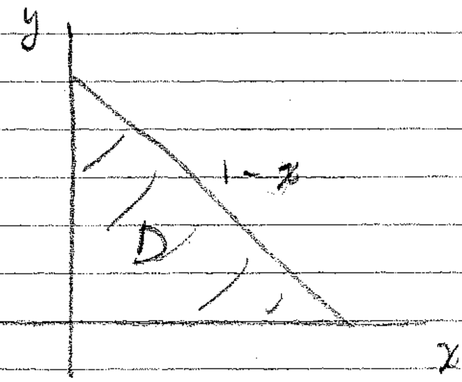
Suppose $E = \{(x,y,z) : x+y+z \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$



$$\iiint_E 1\,dx\,dy\,dz$$

$$= \iint_D \left[\int_0^{1-x-y} dz\right] dx\,dy$$

$$= \iint_D (1-x-y)\,dx\,dy$$



$$= \int_0^1 \left[\int_0^{1-x} (1-x-y)\,dy\right] dx$$

$$= \int_0^1 \left[(1-x)y - \tfrac{1}{2}y^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\,dx = -\tfrac{1}{6}(1-x)^3\Big]_0^1 = \boxed{\tfrac{1}{6}}.$$