Math. 5B. Monday, May 2, 2011.

Gravitation according to Newton:

Force = (mass)(acceleration)

Gravitational Force $= -\dfrac{GMm}{x^2+y^2}\left(\dfrac{x}{\sqrt{x^2+y^2}}\vec{i} + \dfrac{y}{\sqrt{x^2+y^2}}\vec{j}\right)$

$$m\begin{pmatrix} \dfrac{d^2x}{dt^2} \\ \dfrac{d^2y}{dt^2}\end{pmatrix} = -\frac{GMm}{x^2+y^2}\begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}}\end{pmatrix}$$

$$(*) \quad m\frac{d^2\vec{x}}{dt^2} = -\frac{GMm}{|\vec{x}|^2}\left(\frac{\vec{x}}{|\vec{x}|}\right) = -\nabla V$$

How to simplify these equations?

Note $V = -\dfrac{GMm}{|\vec{x}|} = -\dfrac{GMm}{\sqrt{x^2+y^2}}$

Then $\nabla V = +GMm\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}\begin{pmatrix} 2x \\ 2y\end{pmatrix}$ $\quad -\nabla V = -\dfrac{GMm}{x^2+y^2}\begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}}\end{pmatrix}$

$V(x,y)$ = potential energy at (x,y)

Claim: Along a sol'n to $(*)$

$\frac{1}{2}m\left|\frac{d\vec{x}}{dt}\right|^2 + V(x,y)$ = constant

Proof: $\frac{d}{dt}\left[\frac{1}{2}m\left|\frac{d\vec{x}}{dt}\right|^2 + V(x,y)\right]$

$$= m\frac{d\vec{x}}{dt}\cdot\frac{d^2\vec{x}}{dt^2} + \frac{dV}{dt} = m\frac{d^2\vec{x}}{dt^2}\cdot\frac{d\vec{x}}{dt} + \nabla V\cdot\frac{d\vec{x}}{dt} = \left(m\frac{d^2\vec{x}}{dt^2} + \nabla V\right)\cdot\frac{d\vec{x}}{dt} = 0$$

$$\begin{cases} \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] + V(x,y) = E = \text{energy} \\ mx\frac{dy}{dt} - my\frac{dx}{dt} = L = \text{angular momentum (by a similar calculation)}\end{cases}$$

Now transform to polar coordinates using chain rule.

Using techniques of ODE's Newton could then derive Kepler's 3 laws of planetary motion.

<u>Definition</u> A vector field $\vec{F}(x_1,\ldots,x_n)$ is said to be <u>conservative</u> if $\vec{F}(x_1,\ldots,x_n) = \nabla f(x_1,\ldots,x_n)$ for some function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$.

<u>Theorem.</u> Suppose $\vec{x}:[0,1] \longrightarrow \mathbb{R}^n$ is the parametrization of a directed curve $\vec{C}$ from $(a_1,\ldots,a_n)$ to $(b_1,\ldots,b_n)$.

Then $\int_{\vec{C}} \nabla f \cdot \vec{T}\, ds = \int_{\vec{C}} \nabla f \cdot d\vec{x} = f(b_1,\ldots,b_n) - f(a_1,\ldots,a_n)$

Proof: $d\vec{x} = \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$ $\quad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$ $\quad z = f(x_1,\ldots,x_n)$

$z(t) = f(x_1(t),\ldots,x_n(t))$

$$\int_{\vec{C}} \nabla f \cdot d\vec{x} = \int_0^1 \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = \int_0^1 \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$= \int_0^1 \left(\frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} \right) dt = \int_0^1 \frac{d}{dt}(z(t))\, dt = z(1) - z(0)$$

$$= f(b_1,\ldots,b_n) - f(a_1,\ldots,a_n).$$

APPLICATION. Let $\vec{C}$ be the line segment from $(1,3,4)$ to $(3,4,1)$. What is

$\int_{\vec{C}} \nabla f \cdot \vec{T}\, ds$ when $f(x,y,z) = e^{-x^2+y^2+z^2} + x$

SOLUTION: $\int_{\vec{C}} \nabla f \cdot d\vec{x} = f(3,4,1) - f(1,3,4)$

$$= e^{-3^2-4^2-1} + 3 - e^{-1-3^2-4^2} - 1 = \boxed{2}$$

Suppose $\vec{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$

When is $\vec{F}$ conservative? When is $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix} = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

$$\begin{cases} M = \frac{\partial f}{\partial x} \\ N = \frac{\partial f}{\partial y} \end{cases}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

Theorem. If $M(x,y)$ and $N(x,y)$ are well-behaved functions in $\mathbb{R}^2$, then

$$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix} = \nabla f \text{ for some } f \iff \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

Example: $\vec{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

Is F conservative? $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$ NO.

USEFUL NOTATION. If $f(x,y)$ is a smooth function

$$df = \nabla f \cdot d\vec{x} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

CONSIDER THE DE $\frac{dy}{dx} = -\frac{y}{x}$

$x\,dy = -y\,dx \qquad y\,dx + x\,dy = 0$

$\vec{F} \cdot d\vec{x} = 0$, where $\vec{F} = \begin{pmatrix} y \\ x \end{pmatrix}$.

In this case, $\frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}$, so $\vec{F}$ is conservative

$\vec{F} = \nabla f$ where $f(x,y) = xy$

$d(xy) = \frac{\partial (xy)}{\partial x} dx + \frac{\partial (xy)}{\partial y} dy = y\,dx + x\,dy$

DE can be rewritten as $df = 0$, where $f = xy$

GENERAL SOLUTION TO $\frac{dy}{dx} = -\frac{y}{x}$ is $\boxed{xy = C}$

$\vec{F}(x,y) = \begin{pmatrix} \sin y + \cos x \\ (x+1)\cos y \end{pmatrix} = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$

Is $\vec{F}$ conservative? $\frac{\partial N}{\partial x} = \cos y$, $\frac{\partial M}{\partial y} = \cos y$. YES

$\frac{\partial f}{\partial x} = \sin y + \cos x \Rightarrow f = x\sin y + \sin x + g(y)$

$\frac{\partial f}{\partial y} = (x+1)\cos y \Rightarrow f = (x+1)\sin y + h(x)$

$\boxed{f(x,y) = x\sin y + \sin x + \sin y}$