Math. 5B Friday, April 29, 2011

Suppose that $\vec{x} : [a,b] \longrightarrow \mathbb{R}^n$ is a UNIT SPEED parametrization of a directed curve $\vec{C} \subseteq \mathbb{R}^n$

Then $\frac{d\vec{x}}{ds}(s) = \vec{T}(s)$ is a unit-length vector called the unit tangent vector to $\vec{C}$.

If $\vec{x} : [a,b] \longrightarrow \mathbb{R}^n$ is regular but NOT unit speed, then

$$\vec{T}(t) = \frac{\frac{d\vec{x}}{dt}(t)}{\left|\frac{d\vec{x}}{dt}(t)\right|}$$ is the unit tangent vector

Given a directed curve $\vec{C} \subseteq \mathbb{R}^2$ and a smooth vector field

$$\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$$

we can calculate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$

EXAMPLE: $\vec{C}$ = semicircle $x^2 + y^2 = 1$, $y \geq 0$.

$$\vec{F}(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix} = -y\vec{i} + x\vec{j}$$

What is $\int_{\vec{C}} \vec{F} \cdot \vec{T}\, ds$?

$\vec{x} : [0,\pi] \longrightarrow \mathbb{R}^2$ $\quad \vec{x}(s) = \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$

$\vec{T} = \frac{d\vec{x}}{ds} = \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix}$ $\quad \vec{F} \cdot \vec{T} = \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \cdot \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} = \cdots = 1$

$$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds = \int_0^{\pi} 1\,ds = \boxed{\pi}.$$

Although finding the arc length parameter is hard, calculating $\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds$ is usually not, because

$$\vec{T}\,ds = \vec{T}\frac{ds}{dt}\,dt = \frac{\frac{d\vec{x}}{dt}}{\left|\frac{d\vec{x}}{dt}\right|}\frac{ds}{dt}\,dt = \frac{\frac{d\vec{x}}{dt}}{\frac{ds}{dt}}\cdot\frac{ds}{dt}\,dt = \frac{d\vec{x}}{dt}\,dt$$

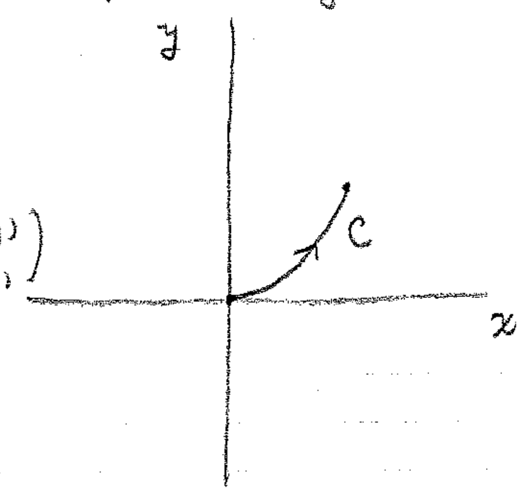
$$= d\vec{x} = dx\,\vec{i} + dy\,\vec{j}.$$

Thus

$$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\vec{C}} \vec{F}\cdot d\vec{x} = \int_{\vec{C}} M\,dx + N\,dy$$

where $\vec{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$

Suppose C is the part of the parabola $y = x^2$ between $(0,0)$ and $(1,1)$ directed to the right.



If $\vec{F}(x,y) = \begin{pmatrix} xy \\ y-3 \end{pmatrix} = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$

$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds$?

$$\int_{\vec{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\vec{C}} M(x,y)\,dx + N(x,y)\,dy.$$

$$= \int_{\vec{C}} xy\,dx + (y-3)\,dy$$

PARAMETRIZATION OF C:

$$\vec{x} : [0,1] \longrightarrow \mathbb{R}^2 \quad \text{by} \quad \vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad \begin{matrix} x = t \\ y = t^2 \end{matrix}$$

$dx = dt \quad dy = 2t\,dt$

$$\int_{\vec{C}} xy\,dx + (y-3)\,dy = \int_0^1 t^3\,dt + (t^2-3)\,2t\,dt$$

$$= \int_0^1 (3t^3 - 6t)\,dt = \left[\frac{3}{4}t^4 - 3t^2\right]_0^1 = \frac{3}{4} - 3 = -\frac{9}{4}$$

<u>Theorem.</u> Suppose $\vec{x} : [0,1] \longrightarrow \mathbb{R}^2$ is a parametrization of a directed curve $\vec{C}$ from (x_0, y_0) to (x_1, y_1)

Then

$$\int_{\vec{C}} \nabla f \cdot \vec{T}\,ds = \int_{\vec{C}} \nabla f \cdot d\vec{x} = f(x_1, y_1) - f(x_0, y_0)$$

Proof:

$$d\vec{x} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} \qquad \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$\nabla f \cdot d\vec{x} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$. Let $z(t) = f(x(t), y(t))$. Then

$$\int_{\vec{C}} \nabla f \cdot d\vec{x} = \int_0^1 \frac{\partial f}{\partial x}\frac{dx}{dt}\,\frac{\partial f}{\partial y}\frac{dy}{dt}\,dt \overset{\text{CHAIN RULE}}{=} \int_0^1 \frac{dz}{dt}\,dt$$

$$= z(1) - z(0) = f(x_1, y_1) - f(x_0, y_0)$$