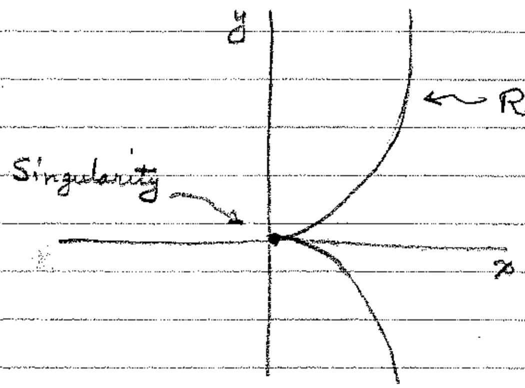
Math. 5B — Wednesday, April 27, 2011

A smooth curve $\vec{x}:[a,b] \longrightarrow \mathbb{R}^N$ is *regular* if $\vec{x}\,'(t)$ is NEVER zero. For example

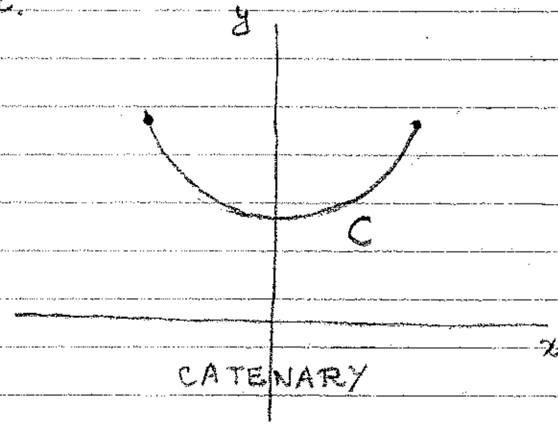
$\vec{x}:\mathbb{R} \longrightarrow \mathbb{R}^2$ $\quad \vec{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ is NOT regular

because $x'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$ and $x'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$\vec{x}$ is NOT regular when $t = 0$

THEOREM. If $\vec{x}:[a,b] \longrightarrow \mathbb{R}^N$ is regular, the $\vec{x}$ can be reparametrized to have unit speed.



Example:

$\vec{x}:[-1,1] \longrightarrow \mathbb{R}^2$ parametrizes C

$\vec{x}(t) = \begin{pmatrix} t \\ \cosh t \end{pmatrix}$

$\vec{x}\,'(t) = \begin{pmatrix} 1 \\ \sinh t \end{pmatrix}$

$|\vec{x}\,'(t)| = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$

$s(t) = \int_0^t |\vec{x}\,'(\tau)|\, d\tau = \int_0^t \cosh \tau \, d\tau = \sinh \tau \Big|_0^t = \sinh t.$

$s = \sinh t \qquad t = \sinh^{-1} s$

$s = \frac{1}{2}(e^t - e^{-t}) \qquad 2s = e^t - e^{-t} \qquad 2se^t = e^{2t} - 1$

$e^{2t} - 2se^t - 1 = 0 \qquad (e^t)^2 - 2s(e^t) - 1 = 0$

$e^t = \dfrac{2s \pm \sqrt{4s^2+4}}{2} = s \pm \sqrt{s^2+1} \qquad e^t > 0 \Rightarrow e^t = s + \sqrt{s^2+1}$

$t = \ln(s + \sqrt{s^2+1})$

$\cosh t = \sqrt{1 + \sinh^2 t} = \sqrt{1+s^2}$

$\vec{x}(s) = \begin{pmatrix} \ln(s+\sqrt{s^2+1}) \\ \sqrt{s^2+1} \end{pmatrix}$ unit speed parametrization of C.

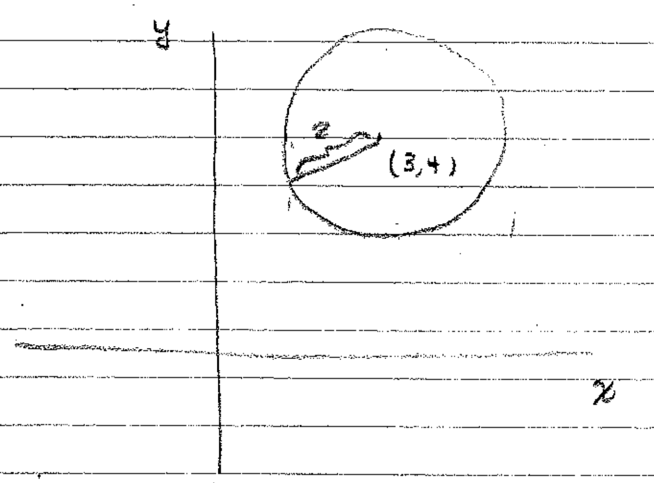
$s \in [-\sinh 1, \sinh 1]$

$$\int_C y^2\,ds = \int_{-\sinh 1}^{\sinh 1} (s^2+1)\,ds = \left[\tfrac{1}{3}s^3 + s\right]_{-\sinh 1}^{\sinh 1}$$

$$= \boxed{\tfrac{2}{3}(\sinh 1)^3 + 2\sinh 1}$$

SOME CURVES ARE EASY TO PARAMETRIZE BY ARC LENGTH.

I. LINE SEGMENTS (LAST TIME)



II. CIRCLES.

$(x-3)^2 + (y-4)^2 = 4 = 2^2$

$\begin{cases} x - 3 = 2\cos t \\ y - 4 = 2\sin t \end{cases}$

$\begin{cases} x = 3 + 2\cos t \\ y = 4 + 2\sin t \end{cases} \qquad \vec{x}(t) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad 0 \le t \le 2\pi$

$$\vec{x}\,'(t) = 2\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \qquad |\vec{x}\,'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

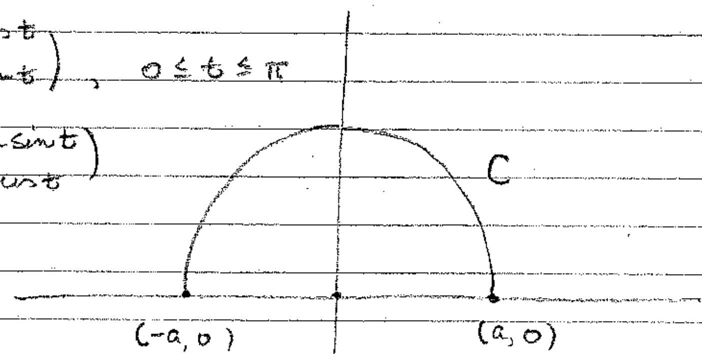
$$\frac{ds}{dt} = |\vec{x}\,'(t)| = 2 \qquad s = 2t \qquad t = \frac{s}{2}$$

$$\vec{x}(s) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} \cos\left(\frac{s}{2}\right) \\ \sin\left(\frac{s}{2}\right) \end{pmatrix} \qquad 0 \le s \le 4\pi$$

CENTER OF MASS OF CIRCULAR WIRE

$$\vec{x}(t) = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}, \quad 0 \le t \le \pi$$

$$\vec{x}\,'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$$



$$|\vec{x}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a \qquad \frac{ds}{dt} = a \qquad s = at \qquad t = \frac{s}{a}$$

$$\vec{x}(s) = \begin{pmatrix} a\cos\left(\frac{s}{a}\right) \\ a\sin\left(\frac{s}{a}\right) \end{pmatrix}, \quad 0 \le s \le \pi a$$

CENTER OF MASS : $(\bar{x}, \bar{y})$, where $\bar{x} = 0$, $\bar{y} = \dfrac{\int_C y\,ds}{\int_C 1\,ds}$

$$\bar{y} = \frac{\int_C y\,ds}{\pi a} = \frac{\int_0^{\pi a} a\sin\left(\frac{s}{a}\right) ds}{\pi a} = \frac{1}{\pi a}\left[-a^2\cos\left(\frac{s}{a}\right)\right]_0^{\pi a}$$

$$= \frac{a}{\pi}\left[-\cos(\pi) + \cos 0\right] = \boxed{\frac{2}{\pi}a}$$

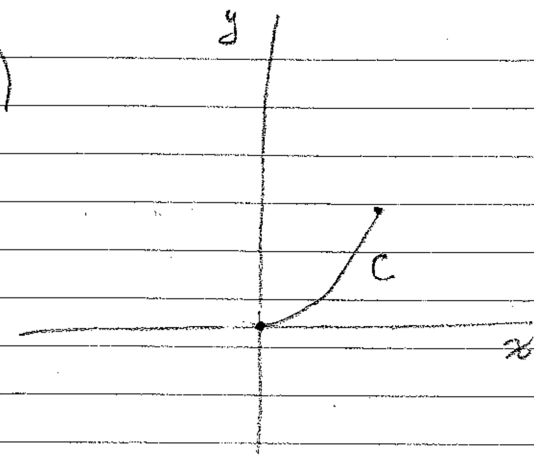
Alternate formula for $\int_C f(x,y)\,ds$.

If $\vec{x} : [a,b] \longrightarrow C$ is a parametrization $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\int_C f(x,y)\,ds = \int_C f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

$\vec{x}:[0,1] \longrightarrow \mathbb{R}^2$ by $\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

parametrized part of a parabola



$\vec{x}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

$\frac{ds}{dt} = |\vec{x}'(t)| = \sqrt{1+4t^2}$

$ds = \sqrt{1+4t^2}\,dt \qquad s = \int_0^t \sqrt{1+4\tau^2}\,d\tau$

DIFFICULT TO INTEGRATE

$\int_C 8x\,ds = ?$

$\int_C 8x\,ds = \int_0^1 8t\sqrt{1+4t^2}\,dt = \frac{2}{3}(1+4t^2)^{\frac{3}{2}}\Big|_0^1$

$= \frac{2}{3}[5^{3/2} - 1] = \frac{2}{3}[5\sqrt{5} - 1]$