Math. 5B Monday, April 25, 2011

If $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a function,

$$\nabla f = \text{gradient of } f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

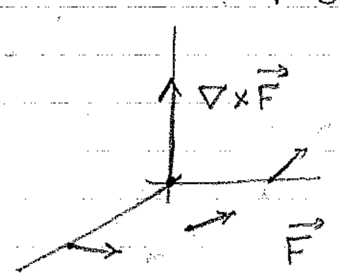
<u>Application</u>. If $\vec{F}(x,y,z)$ is a force field, $\vec{F}$ is <u>conservative</u> if $\vec{F} = -\nabla V$

for some function $V(x,y,z)$ and V is called the <u>potential</u> <u>energy</u> for the force

If $\vec{F}(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$,

$$\text{then } \nabla \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & F_1 \\ \vec{j} & \frac{\partial}{\partial y} & F_2 \\ \vec{k} & \frac{\partial}{\partial z} & F_3 \end{vmatrix}$$

Example: $\vec{F} = \begin{pmatrix} -\frac{1}{4}y \\ \frac{1}{4}x \\ 0 \end{pmatrix}$ $\quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \frac{\partial}{\partial x} & -\frac{1}{4}y \\ \vec{j} & \frac{\partial}{\partial y} & \frac{1}{4}x \\ k & \frac{\partial}{\partial z} & 0 \end{vmatrix} = \frac{1}{2}\vec{k}$



APPLICATION. Suppose $\vec{F} = \rho(x, y, z)\, \vec{V}(x, y, z)$

where ρ = density, $\vec{V}$ = velocity of a steady-state fluid.

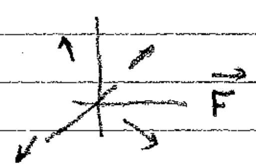
Let $\vec{F} = \rho \vec{V}$

Then $\nabla \times \vec{F}$ = curl $\vec{F}$ = rotation of fluid.

If $\vec{F}(x, y, z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$

then $\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial \vec{F}}{\partial x_2} + \frac{\partial \vec{F}}{\partial x^3}$

Example : $\vec{F}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad \nabla \cdot F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$



$= 3$

Application: If $\vec{F} = \rho \vec{V}$, where ρ = density $\vec{V}$ = velocity

of steady state fluid, then

$\nabla \cdot (\rho \vec{V})$ = rate of creation − destruction of fluid.

IDENTITIES:

$\nabla \times (\nabla f) = \vec{0} \qquad \nabla \cdot (\nabla \times \vec{F}) = 0$

If $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is globally well-behaved,

$\nabla \times \vec{F} = \vec{0} \Rightarrow \vec{F} = \nabla f$ for some f

UNDERSTANDING OF grad, div, curl comes from line surface and volume integrals.

LINE INTEGRALS

$ds^2 = dx^2 + dy^2$ in $\mathbb{R}^2$ $\qquad ds^2 = dx^2 + dy^2 + dz^2$ in $\mathbb{R}^3$.

Suppose $\vec{x} : [a,b] \longrightarrow \mathbb{R}^2$ is a parametrization of a curve C in $\mathbb{R}^2$

$L(C)$ = length of C = $\int_a^b$ speed dt $= \int_a^b |\vec{x}'(t)|\,dt = \int_a^b ds$

$ds = |\vec{x}'(t)|\,dt$.

Example: $\vec{x}(t) = \begin{pmatrix} t \\ \cosh t \end{pmatrix}$, $0 \le t \le t_0$ $\qquad \cosh t = \frac{1}{2}(e^t + e^{-t})$

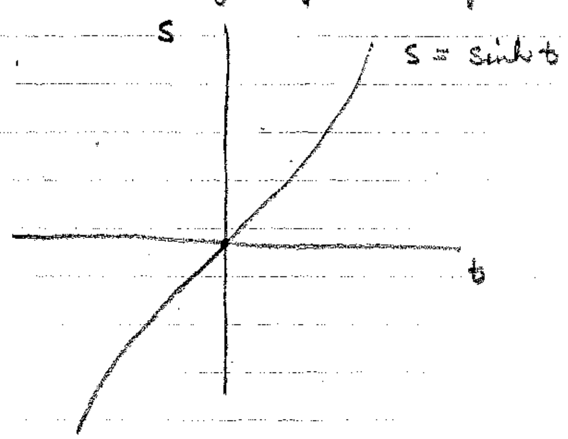
$\vec{x}'(t) = \begin{pmatrix} 1 \\ \sinh t \end{pmatrix}$ $\qquad$ CATENARY $\qquad \sinh t = \frac{1}{2}(e^t - e^{-t})$

$\cosh^2 t - \sinh^2 t = 1$

$|\vec{x}'(t)| = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$

$L(C) = \int_0^{t_0} |x'(t)|\,dt = \int_0^{t_0} \cosh t\,dt = \sinh t \Big|_0^{t_0} = \sinh t_0$.

$s_0 = s(t_0)$ = length of curve from 0 to t_0 = $\sinh t_0$



$s = \frac{1}{2}(e^t - e^{-t})$

$2s = e^t - e^{-t}$

$2se^t = e^{2t} - 1$

$e^{2t} - 2se^t - 1 = 0$

$e^t = \dfrac{2s \pm \sqrt{4s^2+4}}{2}$

$t = \ln(s + \sqrt{s^2+1})$