Math. 5B Wednesday, April 20, 2011

Suppose S is a surface in $\mathbb{R}^3$ defined by $\phi(x,y,z) = 0$

where ϕ is a smooth real-valued function.

Suppose $\nabla\phi \neq 0$ at any point of S.

If $(x_0, y_0, z_0) \in S$, then $\nabla\phi(x_0, y_0, z_0)$ is $\perp$ to S at (x_0, y_0, z_0).

Equation of tangent plane at (x_0, y_0, z_0):

$$(\nabla\phi)(x_0, y_0, z_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$$

Example: Find an equation for the plane tangent to

$$x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 3 \quad \text{at} \quad (1, 2, 2)$$

$$\phi(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{4} - 3$$

$$\nabla\phi = \left(2x, \frac{y}{2}, \frac{z}{2}\right) \qquad \nabla\phi(1,2,2) = (2, 1, 1)$$

Equation of plane: $(2, 1, 1) \cdot (x-1, y-2, z-2) = 0$.

$2(x-1) + (y-2) + (z-2) = 0$.

Suppose now that S is a surface defined by $\phi(x,y,z) = 0$

Find maximum and minimum values of f on S.

At a max. or min (x_0,y_0,z_0), $\nabla f(x_0,y_0,z_0)$ must be a multiple of $\nabla\phi(x_0,y_0,z_0)$. Need to solve

$$* \begin{cases} \nabla f(x_0,y_0,z_0) = \lambda_0 \nabla\phi(x_0,y_0,z_0) \\ \phi(x_0,y_0,z_0) = 0 \end{cases}$$

This is called the method of Lagrange multipliers

$$\begin{cases} H(x,y,z,\lambda) = f(x,y,z) - \lambda\,\phi(x,y,z) \\ \dfrac{\partial H}{\partial x} = \dfrac{\partial f}{\partial x} - \lambda\dfrac{\partial \phi}{\partial x} = 0 \\ \dfrac{\partial H}{\partial y} = \dfrac{\partial f}{\partial y} - \lambda\dfrac{\partial \phi}{\partial y} = 0 \\ \dfrac{\partial H}{\partial z} = \dfrac{\partial f}{\partial z} - \lambda\dfrac{\partial \phi}{\partial z} = 0 \\ \dfrac{\partial H}{\partial \lambda} = -\phi = 0 \end{cases}$$

Solutions to $(*)$ are critical points for H.

Example : $f(x,y,z) = 4x + 4y + 6z$

S : $x^2 + 2y^2 + 3z^2 = 1$ $\qquad \phi(x,y,z) = x^2 + 2y^2 + 3z^2 - 1$

$$H(x,y,z,\lambda) = 4x + 4y + 6z - \lambda(x^2 + 2y^2 + 3z^2 - 1)$$

$$\frac{\partial H}{\partial x} = 4 - 2x\lambda = 0 \qquad x = \frac{2}{\lambda}$$

$$\frac{\partial H}{\partial y} = 4 - 4y\lambda = 0 \qquad y = \frac{1}{\lambda}$$

$$\frac{\partial H}{\partial z} = 6 - 6z\lambda = 0 \qquad z = \frac{1}{\lambda}$$

$$\phi(x,y,z) = x^2 + 2y^2 + 3z^2 - 1 = \frac{4}{\lambda^2} + \frac{2}{\lambda^2} + \frac{3}{\lambda^2} - 1$$

$$\frac{9}{\lambda^2} = 1 \qquad \lambda^2 = 9 \qquad \lambda = \pm 3$$

$$\begin{cases} x = \frac{2}{3} \\ y = \frac{1}{3} \\ z = \frac{1}{3} \end{cases} \qquad f(x,y,z) = 4 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = 6$$

$$\begin{cases} x = -\frac{2}{3} \\ y = -\frac{1}{3} \\ z = -\frac{1}{3} \end{cases} \qquad f(x,y,z) = -6$$