Math 5B Monday, April 18, 2011

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x^n} \end{pmatrix} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$\vec{c} = (c_1, \ldots, c_n)$ is a <u>critical point</u> for f if $\nabla f(\vec{c}) = 0$

Example: $f(x,y) = \frac{1}{2} y^2 - \cos x$

$$\nabla f = \begin{pmatrix} \sin x \\ y \end{pmatrix} = \vec{0} \Longrightarrow \begin{cases} \sin x = 0 \\ y = 0 \end{cases}$$

$\sin x = 0 \Rightarrow x = n\pi$, where $n \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

CRITICAL POINTS ARE CANDIDATES FOR MAXIMA AND MINIMA.

SECOND DERIVATIVE TEST ?

If $\vec{c} = (c_1, \ldots, c_n)$ is a critical point for f, the <u>Hessian</u> matrix at $\vec{c}$ is

$$A = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) & \cdots & \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial x_1}\right) \\ \vdots & & \\ \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_n}\right) & \cdots & \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial x_n}\right) \end{pmatrix}(\vec{c})$$

Example: If $f(x,y) = \frac{1}{2}y^2 - \cos x$

$$\nabla f = \begin{pmatrix} -\sin x \\ y \end{pmatrix}$$

Critical points $(n\pi, 0)$, $n \in \mathbb{Z}$

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}(n\pi, 0) = \begin{pmatrix} -\cos x & 0 \\ 0 & 1 \end{pmatrix}(n\pi, 0)$$

$$= \begin{pmatrix} (-1)^{n} & 0 \\ 0 & 1 \end{pmatrix}$$

Equality of mixed partials $\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \Rightarrow A = A^T$

THEOREM. If A is an $n \times n$ matrix such that $A^T = A$, then A has n REAL eigenvalues $\lambda_1, \dots, \lambda_n$, counted with multiplicity. Moreover $\exists$ an $n \times n$ matrix B such that $B^T = B^{-1}$ and

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

SECOND DERIVATIVE TEST:

Let $\vec{c}$ be a critical point for f, A its Hessian matrix at $\vec{c}$.

If $\det A \neq 0$, $\vec{c}$ is a <u>nondegenerate</u> critical point.

Otherwise it is <u>degenerate</u>.

Suppose $\vec{c}$ is a nondegenerate critical point

$\vec{c}$ is a <u>local minimum</u> if eigenvalues of A are positive

$\vec{c}$ " " <u>local maximum</u> if " " " " negative.

$\vec{c}$ is a <u>saddle point</u> of index k, $0 < k < n$

if k eigenvalues are < 0
$n-k$ eigenvalues are > 0

Example: $f(x, y) = \frac{1}{2} y^2 - \cos x$

Critical points: $(n\pi, 0)$, $n \in \mathbb{Z}$

$$A = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}$$

$(n\pi, 0)$ is a local minimum when n even

$(n\pi, 0)$ is a saddle point when n odd.

EXAMPLE: $f(x,y) = 2x^2 + 5xy + 2y^2 - 4x - 5y$

CRITICAL POINTS ?

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x + 5y - 4 = 0 \\ \dfrac{\partial f}{\partial y} = 5x + 5y - 5 = 0 \end{cases}$$

$$\begin{cases} 4x + 5y = 4 \\ 5x + 4y = 5 \end{cases}$$

CRAMER'S RULE: $x = \dfrac{\begin{vmatrix} 4 & 5 \\ 5 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & 5 \\ 5 & 4 \end{vmatrix}} = 1$

$$y = \dfrac{\begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 5 \\ 5 & 4 \end{vmatrix}} = 0$$

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

Eigenvalues ?

$$\begin{vmatrix} 4-\lambda & 5 \\ 5 & 4-\lambda \end{vmatrix} = 0 \qquad (4-\lambda)^2 - 5^2 = 0 \qquad \lambda - 4 = \pm 5$$

$$\lambda = 9 \text{ OR } -1$$

$\therefore$ $(1, 0)$ is a saddle point.