Math 3C Lecture 4

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PICARD'S THEOREM. Suppose that the real-valued function $f(x, y)$ is well-behaved throughout the region D in the (x, y)-plane. Then there is a unique maximal solution curve for the differential equation

$$\frac{d}{dx}(y(x)) = f(x, y(x))$$

which passes through every point (x_0, y_0) of D.

(A "maximal" solution curve is one that cannot be extended without running out of D.)

A differential equation is in **standard form** if it is written in the form

$$\frac{dy}{dx} = f(x, y)$$

It is important to recognize that not all differential equations have solutions, and some differential equations have more than one. For example, the differential equation

$$\left(\frac{dy}{dx}\right)^2 + y^2 + x^2 = -1$$

has no solutions. On the other hand, the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 1 - y^2$$

does have solutions in part of the (x, y)-plane. But to apply Picard's Theorem to this equation, we need to **solve for** dy/dx and put the differential equation into **standard form**:

$$\frac{dy}{dx} = f(x, y)$$

If we do this we obtain

$$\frac{dy}{dx} = \pm\sqrt{1 - y^2}.$$

This can be thought of as two separate differential equations:

$$\frac{dy}{dx} = f_1(x, y) = \sqrt{1 - y^2}$$

and

$$\frac{dy}{dx} = f_2(x, y) = -\sqrt{1 - y^2}$$

The functions $f_1(x, y)$ and $f_2(x, y)$ are well-behaved in the region

$$D = \{t, y) : -1 < y < y\}.$$

Each of them should have exactly one solution curve through each point.

We can solve

$$\frac{dy}{dx} = \pm\sqrt{1-y^2}$$

by separation of variables:

$$\frac{dy}{\pm\sqrt{1-y^2}} = dx.$$

To integrate on the left, we use trigonometric substitution.

As another example, we can consider the differential equation

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

We can write this as

$$\frac{dy}{dx} = f(x, y)$$

where

$$f(x, y) = \left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

In this case, $f(x, y)$ is well-behaved in the region

$$D = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}.$$

Thus we should expect exactly one maximal solution to this differential equation passing through each point of this region.

The differential equation

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

cannot be solved directly by separation of variables. However, suppose that we try the substitution

$$z = \frac{y}{x}.$$

Then

$$y = xz \quad \Rightarrow \quad \frac{dy}{dx} = z + x\frac{dz}{dx},$$

so the differential equation becomes

The general solution to our differential equation is

$$y = \frac{-x}{\log|x| + c}.$$

No matter what c is, the resulting solution curve goes through the point $(0, 0)$. On the other hand there are no solution curves through any points $(0, y_0)$ for $y_0 \neq 0$. This was to be expected because the function

$$f(x, y) = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

appearing in the differential equation, is **not well-behaved** when $x = 0$.

ORTHOGONAL TRAJECTORIES. Recall that if

$$\frac{dy}{dx} = f(x, y),$$

then

$$f(x, y) = \text{slope of solution curve.}$$

Thus if θ is the angle that the solution curve at (x_0, y_0) makes with the x-axis,

$$f(x, y) = \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

A curve perpendicular to the solution curve would have slope

$$\begin{aligned} \tan(\theta + \pi/2) &= \frac{\sin(\theta + \pi/2)}{\cos(\theta + \pi/2)} \\ &= \frac{\cos\theta}{-\sin\theta} = \frac{-1}{\tan\theta} = \frac{-1}{f(x, y)}. \end{aligned}$$

Thus the solution curve to

$$\frac{dy}{dx} = \frac{-1}{f(x,y)}$$

which passes through the point (x_0, y_0) will be orthogonal to the solution curve to

$$\frac{dy}{dx} = f(x,y)$$

which passes through the same point.

If a family of curves is represented by the differential equation

$$\frac{dy}{dx} = f(x,y)$$

then its **orthogonal trajectories** are the curves which solve the differential equation

$$\frac{dy}{dx} = \frac{-1}{f(x,y)}$$

For example, the family of curves

$$y = \frac{1}{2}x^2 + c$$

is the set of solution curves to the differential equation

$$\frac{dy}{dx} = x.$$

Thus the family of **orthogonal trajectories** will be the set of solution curves to the differential equation

$$\frac{dy}{dx} = \frac{-1}{x}.$$