Math 3CI: Project 8
Basic linear algebra ideas

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Earlier we studied homogeneous linear differential equations, such as

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0. \tag{1}$$

and we showed that if $y_1(t)$ and $y_2(t)$ are solutions to this differential equation, then so $y_1(t) + y_2(t)$ and $cy_1(t)$ for any real number c. This motivates the following definition:

Definition. Suppose that V is a nonempty set of functions from $\mathbb{R}$ to $\mathbb{R}$ and that whenever $y_1(t)$ and $y_2(t)$ are elements of V, then so is $y_1(t) + y_2(t)$ and $cy_1(t)$ for any real number c. Then we say that V is a *vector space*, or since it is a space of functions, a *linear function space*.

Thus we can say that the space V of solutions to (1) is a vector space.

Definition. Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$. Then the set of all constant multiples of f is called the set of *linear combinations* of f or the *span* of f. If f and g are functions, then the set

$$\mathrm{Span}(f, g) = \{c_1 f + c_2 g : c_1, c_2 \in \mathbb{R}\}$$

is called the set of *linear combinations* of $\{f, g\}$ or the *span* of $\{f, g\}$. If f, g and h are functions, then the set

$$\mathrm{Span}(f, g, h) = \{c_1 f + c_2 g + c_3 h : c_1, c_2, c_3 \in \mathbb{R}\}$$

is called the set of *linear combinations* of $\{f, g, h\}$ or the *span* of $\{f, g, h\}$. I bet you can guess what the span of a collection of n functions is.

Note that if f, g and h are functions from $\mathbb{R}$ to $\mathbb{R}$, then

$$V = \mathrm{Span}(f, g, h)$$

is a vector space.

1.a. Suppose that f, g and h are functions, none of which are zero. Show that if $f + g + h = 0$, then

$$\text{Span}(f, g, h) = \text{Span}(f, g) = \text{Span}(f, h) = \text{Span}(g, h).$$

b. Suppose that f, g and h are functions, none of which are zero. Show that if $3f + 7g - 4h = 0$, then

$$\text{Span}(f, g, h) = \text{Span}(f, g) = \text{Span}(f, h) = \text{Span}(g, h).$$

c. Suppose that f, g and h are functions, none of which are zero. Show that if $c_1 f + c_2 g + c_3 h = 0$, where c_1, c_2 and c_3 are nonzero real numbers, then

$$\text{Span}(f, g, h) = \text{Span}(f, g) = \text{Span}(f, h) = \text{Span}(g, h).$$

Definition. Suppose that $f_1, \cdots, f_n$ are n functions from $\mathbb{R}$ to $\mathbb{R}$. We say that these functions are *linearly dependent* over $\mathbb{R}$ if there exist real numbers $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0.$$

We say that a collection $\{f_1, f_2, \ldots, f_n\}$ of functions are *linearly independent* over $\mathbb{R}$ if they are not linearly dependent.

2. Which of the following collections of functions are linearly dependent?

a. $\{1, t, t^2, t^3\}$.

b. $\{1, t, t^2, t^3 + t^2 + t + 1\}$.

c. $\{1, t, t^2, t^3, t^3 + t^2 + t + 1\}$.

d. $\{t, t + 7, t + 3, t^2\}$.

3. Generalize the ideas in problem 1 and show that if a collection $\{f_1, f_2, \ldots, f_n\}$ of functions is linearly dependent, then their span is the same as the span of fewer of these functions.

4. Generalize the ideas in problem 2 and show that if a collection of n polynomials has different degrees, then the collection of polynomials is linearly independent.

5. Suppose that you have a finite set of polynomials. Can you always find a set of polynomials of different degrees with the same span? Is the resulting collection of polynomials (of different degrees) linearly independent? Explain your reasoning.

Definition. Suppose that V is a vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. A collection of functions $\{f_1, f_2 \dots f_n\}$ is a *basis* for V over $\mathbb{R}$ if

1. $V = \text{Span}(f_1, f_2, \dots, f_n)$, and
2. $\{f_1, f_2, \dots, f_m\}$ is linearly independent.

6.a. Is $\{e^t, e^{-t}\}$ a basis for $\text{Span}(e^t, e^{-t})$? Why or why not?

b. Is $\{e^t, e^{-t}, e^t + 2e^{-t}\}$ a basis for $\text{Span}(e^t, e^{-t})$? Why or why not?

c. Is $\{\cos t, \sin t\}$ a basis for $\text{Span}(\cos t, \sin t)$? Why or why not?

7.a. Suppose that V is the space of solutions to the homogeneous linear differential equation

$$\frac{d^2y}{dt^2} - y = 0.$$

It follows from the theory of homogeneous linear differential equations that V is a vector space of functions. Find a basis for V. In other words, find a collection of functions $\{f, g\}$ such that $V = \text{Span}(f, g)$ and $\{f, g\}$ are linearly independent.

b. Suppose that V is the space of solutions to the homogeneous linear differential equation

$$\frac{d^2y}{dt^2} + y = 0.$$

Find a basis for V.

c. Suppose that V is the space of solutions to the homogeneous linear differential equation

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = 0.$$

Find a basis for V. (Hint: For elements of the basis, try functions of the form $t^k e^t$.)

d. Suppose that V is the space of solutions to the homogeneous linear differential equation

$$\frac{d^4y}{dt^4} - 2\frac{d^2y}{dt^2} + y = 0.$$

Find a basis for V.