Math 3CI: Project 7
The motion of a cart attached to the wall by a spring
(revised version)

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We consider the problem of a cart moving along a track attached to a wall by means of a spring. We let

$$x(t) = \text{position of the cart to the right of equilibrium at time } t.$$

We suppose that there are three forces acting on the cart. First, a spring force

$$F_{\text{spring}} = -kx,$$

where k is a positive constant, called the spring constant. Second, a linear damping force,

$$F_{\text{damping}} = -c\frac{dx}{dt},$$

where c is a positive constant, the damping force being assumed to be linear in the velocity and in the opposite direction to the motion of the cart. Finally, we suppose that there is an external force

$$F_{\text{external}} = f(t),$$

where $f(t)$ is a function that might be given, say $f(t) = \sin(3t)$, for example. We would like to determine the motion of the cart.

In accordance with Newton's law of motion, the total force must satisfy the equation

$$F_{\text{total}} = m\frac{d^2x}{dt^2},$$

where m is the mass of the cart, and hence

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx + f(t).$$

We thus obtain a second order linear differential equation with constant coefficients:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t), \tag{1}$$

where the constants m, c, and k are positive. Note that this is a second-order linear differential equation, nonhomogeneous if $f(t)$ is nonzero.

1. One way of approaching the differential equation (1) is by "reduction of order." That is, we figure that it is easier to use first-order derivatives rather than second order derivatives, so we set $y = dx/dt$ and rewrite (1) as a first-order system

$$\frac{dx}{dt} = \cdots, \qquad \frac{dy}{dt} = \cdots. \tag{2}$$

What is the first-order system of differential equations that you obtain by this process?

2. Let h be a small real number, for example $h = .01$. You would like to write out a system difference equations that can give approximate solutions to the system (2). Recalling that

$$\frac{x(t+h) - x(t)}{h} \quad \text{is an approximation to} \quad \frac{dx}{dt}(t)$$

and

$$\frac{y(t+h) - y(t)}{h} \quad \text{is an approximation to} \quad \frac{dy}{dt}(t)$$

write out a system of difference equations

$$x(t+h) = x(t) + h(\cdots), \qquad y(t+h) = y(t) + h(\cdots)$$

which should give approximations to the solutions to the system (2). (This is called the *Cauchy-Euler method* for finding numerical solutions to the system of differential equations.)

3. It is often expedient to approach the motion of the cart by dividing the solution of (1) into two steps. The first step is to understand the "free motion" of the cart, that is, the motion when the external force $f(t)$ is zero. This corresponds to the *associated homogeneous linear differential equation*

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \tag{3}$$

The idea is to find the general solution in this case, and then add the complication of an external force in the second step. In this case, reduction of order yields

$$\begin{aligned} \tfrac{dx}{dt} &= \qquad\qquad y, \\ \tfrac{dy}{dt} &= -\tfrac{k}{m}x \quad -\tfrac{c}{m}y. \end{aligned} \tag{4}$$

You can eliminate t from the resulting equations obtaining an equation involving only x and y. What is that equation? Using the substitution $z = y/x$, you can separate variables. Carry out the separation of variables, but do not try to do the integration. (For various choices of c, k and m you could carry out the integrations. But you would soon find out that there are faster methods.)

4. Find the general solution to (3) in the special case where $m = 1$, $c = 4$ and $k = 3$. (Hint: Find two "independent" solutions and use superposition.) What is the motion of the cart when the position at time zero is $x(0) = 0$ and the velocity at time zero is $y(0) = (dx/dt)(0) = 5$?

5. Find the general solution to (3) in the case where $c = 0$. (Hint: Previous experience suggests that the solution should be a superposition of sines and cosines. Thus you might try $x(t) = \cos(\omega t)$ or $x(t) = \sin(\omega t)$, and solve for ω.) What is the frequency of oscillation of the cart as a function of m and k? What happens to the frequency if you double the spring constant k?

6. What is the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0?$$

What is the general solution to the nonhomogeneous differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = \sin(3t)?$$

7. We can mathematical software to study the qualitative motion of the cart as the linear damping increases. Suppose that $m = k = 1$. Then the system (4) becomes

$$\begin{aligned} \frac{dx}{dt} &= && y, \\ \frac{dy}{dt} &= -x && -cy. \end{aligned}$$

Use mathematical software available on the web to study the orbits of this system in the "phase plane," that is the (x, y)-plane, for various choices of c, say $c = 0$, $c = .1$, $c = .2$, $c = .5$ and $c = 1$. The "direction field" for the system of differential equations can be sketched by software (PPLANE 2005.10) available at:

http://math.rice.edu/∼dfield/dfpp.html

By clicking at a point in the "phase plane" window, you can have the software sketch a solution curve which starts at that point. Print out some representative orbits to include in your notebook.

8. Find the general solution to

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0.$$

(This problem is a bit harder than the previous ones if you don't use complex numbers. If you want to avoid complex numbers, you might want to try solutions of the form $x(t) = e^{\lambda t}\cos(\omega t)$.) What is the general solution to

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-.5t}.$$

9. Find the general solution to the third order differential equation

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0.$$

What is the solution to the initial value problem

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1, \quad \frac{d^2x}{dt^2}(0) = 0?$$

10. Find the general solution to the differential equation

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0.$$

(The difficulty here is that the roots of the characteristic equation are repeated. You might try a solution of the form $x(t) = te^{\lambda t}$, and solve for λ.) What is the solution to the initial value problem

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1?$$

11. Find the general solution to the system of differential equations

$$\begin{aligned} \tfrac{dx}{dt} &= x - 3y, \\ \tfrac{dy}{dt} &= 2x - 4y. \end{aligned} \tag{5}$$

(Hint: You might try to eliminate one of the dependent variables. Systems like this but with many more variables, come up when considering configurations of weights connected with springs. We need to develop some techniques to find the solutions efficiently.)