Math 3CI: Project 6A
Linear differential equations

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Linear differential equations are differential equations that look like this:

$$\frac{dy}{dt} + p_0(t)y = f(t),$$

$$\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t),$$

$$\frac{d^3y}{dt^3} + p_2(t)\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t),$$

and so forth. Of course, they can have even more derivatives. Note that the functions $p_i(t)$ and $f(t)$ are functions of only the single variable t. These equations are said to be *homogeneous* if $f(t) = 0$.

Earlier, you have found some solutions to linear differential equations such as the equation

$$\frac{dy}{dt} - ky = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} + y = 0$$

by various ad hoc techniques. We would like to develop more systematic approaches to solving linear differential equations.

1. Show the following: If $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous linear differential equation

$$\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = 0, \tag{1}$$

then $y_1(t) + y_2(t)$ is also a solution. If $y_1(t)$ is a solution to (1) and c is a constant, then $cy_1(t)$ is also a solution. If $y_1(t)$ and $y_2(t)$ are solutions to (1), then so is an arbitrary superposition $c_1y_1(t) + c_2y_2(t)$. If $y_1(t)$ is a solution to (1) and $y_2(t)$ is a solution to

$$\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_0(t)y = f(t), \tag{2}$$

then $y_1(t) + y_2(t)$ is a solution to (2).

2. Check that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are solutions to the differential equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + y = 0.$$

From this information, what is the most general possible solution to the differential equation of simple harmonic motion? Can you find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + y = 3e^t?$$

Hint: First find a *particular* solution of the form $y(t) = Ae^t$, and then use linearity. Can you find the general solution to

$$\frac{d^2y}{dt^2} + y = 3t^2 + 2t - 5?$$

3. Next, consider the differential equation

$$\frac{dy}{dt} + p(t)y = f(t). \tag{3}$$

We say that

$$\frac{dy}{dt} + p(t)y = 0. \tag{4}$$

is the associated first order differential equation. Show that the associated first order equation can always be solved by separation of variables. Find the general solution to the differential equation

$$\frac{dy}{dt} + \frac{1}{t+1}y = 0,$$

and write it in the form

$$y(t) = cy_1(t), \quad \text{where } y_1(t) \text{ is a nonzero particular solution to (4).}$$

Here is a trick that can be used to solve (3) when we know a particular solution $y_1(t)$ to (4). Set $y(t) = v(t)y_1(t)$, substitute into (3) and solve for $v(t)$. Use this idea to find the general solution to

$$\frac{dy}{dt} + \frac{1}{t+1}y = \frac{t+2}{t+1}.$$

4. Use the method described above to find the general solution to the differential equation

$$\frac{dy}{dt} - \frac{1}{t+1}y = t^2, \quad t \geq 0.$$

5. Can you find any solutions of the form $y(t) = e^{\lambda t}$, where λ is constant, to the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0?$$

What is the general solution to the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0?$$

What is the general solution to the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}?$$

6. Can you find any solutions of the form $y(t) = t^\lambda$, where λ is constant, to the differential equation

$$t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0?$$

Note that you can divide by t^2 and rewrite this equation as

$$\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0,$$

so that it is a linear homogeneous differential equation of the form (2) with $p_1(t) = -2/t$ and $p_2(t) = 2/t^2$. What is the general solution to the differential equation

$$t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0?$$