

Math 3CI: Project 5A
Techniques for finding explicit solutions to
differential equations (revised version)

October 16, 2009

We want to consider some further techniques for solving differential equations.

1. Let's start with the relatively simple differential equation

$$\frac{dy}{dt} = -2ty.$$

It looks like this differential equation could be solved by separating variables, just like the equation of exponential growth and the logistic equation. Find all possible solutions to this differential equation. What you should obtain is the “general solution” containing one constant of integration that can be determined by an initial condition. Sketch the solutions in the (t, y) -plane. There should be exactly one solution $y = y(t)$ which passes through each point of the (t, y) -plane. (Why?) Does each initial-value problem

$$\frac{dy}{dt} = -2ty, \quad y(0) = c$$

have a solution which is well-behaved for all positive time?

2. Next consider the differential equation

$$\frac{dy}{dt} = y^2.$$

Again, it looks like this differential equation might be solved by separating variables. Find all possible solutions to this differential equation. Sketch the solutions in the (t, y) -plane. Again there should be exactly one solution $y = y(t)$ which passes through each point of the (t, y) -plane. Does each initial-value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = c$$

have a solution which is well-behaved for all positive time? Given $y(0)$, can you tell how long it takes for $y(t)$ to “blow up”?

3. Next, consider the differential equation

$$\frac{dy}{dt} = \left(\frac{y}{t}\right)^2 + \left(\frac{y}{t}\right).$$

Note that the right-hand side is not well-behaved when $t = 0$, so we might expect the solutions to exhibit some problems when $t = 0$. You can try, but you probably will have difficulty separating variables this time. How about trying a substitution, say

$$z = \frac{y}{t}, \quad \text{or} \quad y = tz.$$

Does the equation simplify if you substitute in $y = tz$ and then eliminate y ? Can you use this substitution to find the “general solution”?

4. The differential equation

$$\frac{d^2x}{dt^2} + x = 0 \tag{1}$$

is said to be *second order* because it involves a second derivative. One approach to solving a differential equation like this is to reduce it to a first order system. If you set

$$y = \frac{dx}{dt}, \quad \text{then} \quad \frac{dy}{dt} = \frac{d^2x}{dt^2} = -x,$$

so the second-order differential equation is equivalent to the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x. \end{aligned} \tag{2}$$

4a. Eliminate t between the two equations in the system to obtain a differential equation involving only x and y . Find the general solution to this differential equation. The general solution should involve one constant of integration. Can you graph the trajectories of the solutions to (2)?

4b. You can eliminate y from the resulting general solution, by using the fact that $y = dx/dt$. You thereby obtain a differential equation involving only t and x . Find the general solution to this differential equation. The general solution should involve two constants of integration. Can you graph the resulting functions of time?

5. The differential equation

$$\frac{d^2x}{dt^2} + \sin x = 0 \tag{3}$$

is called the *pendulum equation*. If you set

$$y = \frac{dx}{dt}, \quad \text{then} \quad \frac{dy}{dt} = \frac{d^2x}{dt^2} = -\sin x,$$

so the second-order differential equation is equivalent to the first-order system

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\sin x.\end{aligned}\tag{4}$$

Eliminate t from these equations and solve to find a relationship between x and y . Sketch the resulting curves in the (x, y) -plane. Using the fact that $y = dx/dt$, you should be able to derive a differential equation that involves just x and t . But don't try to do the integration—it would require new functions (elliptic functions).

A solution $x = x(t)$ to a differential equation such as (1) or (3) is said to be *periodic* if there is some constant T such that $x(t+T) = x(t)$. Are the solutions to (1) periodic? What about the solutions to (3)?