Math 3CI: Project 4
The logistic equation

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In Project 2, we modeled population growth under the assumptions that the instantaneous birth rates and death rates were constant. If $y(t)$ is population at time t, where t is measured in years past the present, those assumptions lead to the differential equation

$$\frac{dy}{dt} = ky, \quad \text{where} \quad k = (\text{instantaneous birth rate}) - (\text{instantaneous death rate}).$$

You were able to solve this differential equation by the process of separation of variables,

$$\frac{dy}{y} = kdt, \quad \log|y| = kt + c_1, \quad |y| = e^{kt+c_1}, \quad y = (\pm e^{c_1})e^{kt}, \quad y = ce^{kt}, \quad (1)$$

where c is a constant of integration.

However, the equation of exponential growth leads to unrealistic conclusions.

1. The current population of the earth is 6.7×10^9 and the population growth rate for the earth is currently 1.2% per year, corresponding to an approximate instantaneous growth rate of $k = .012$. Assuming that this growth rate remains constant, how long will it be before the population density equals one person per square foot of the earth's surface? (Hint: To answer this question, we need to determine the area of the earth in square feet. Assuming the radius r is

$$4000\text{miles} = (4000)(5280)\text{feet},$$

we can plug into the formula Area $= 4\pi r^2$ to obtain

$$\text{Surface area of the earth} = 4\pi(4000)^2(5280)^2\text{ft}^2 \doteq 5.6 \times 10^{15}.)$$

2. You decide that assuming birth and death rates are constant is clearly unrealistic, so you decide to utilize a model which assumes that death rate goes up as population increases. This leads you to the differential equation

$$\frac{dy}{dt} = ky - ay^2, \quad (2)$$

where k and a are positive constants. Suppose you use this to model human population on the earth. What would a need to be in order for the earth's population to stabilize at 15 billion people?

3. The differential equation (2) is called the *logistic equation*. Can you determine the general solution to this equation? You might want to follow the model given in (1). Your solution should involve one constant of integration.

4. Looked at simply as a differential equation, the logistic equation (2) has two constant solutions, one being the solution $y = 0$. Is the other constant solution stable?

5. Suppose that you wanted to find a numerical approximation to the logistic differential equation with initial conditions

$$\frac{dy}{dt} = .01y - 10^{-12}y^2, \quad y(0) = 6.7 \times 10^9. \tag{3}$$

Can you determine the corresponding difference equation

$$y(t+1) = y(t) + \cdots ?$$

Suppose you decide to approximate the derivative

$$\frac{dy}{dt} \quad \text{by} \quad \frac{y(t+h) - y(t)}{h},$$

where $h = .1$. Can you determine the corresponding difference equation

$$y(t+h) = y(t) + \cdots ?$$

Set up an Excel spreadsheet to determine the numerical solution to (3) when you use the derivative approximation with $h = .1$ and carry the calculation out for 15 years into the future.

6. Suppose we use the logistic equation (2) to model the number of fish in a pond. Suppose that time t is measured in years and the fish population in three succeeding years is measured to be

$$y(0) = 2{,}000{,}000, \qquad y(1) = 4{,}000{,}000, \qquad y(2) = 6{,}000{,}000.$$

What is the equilibrium value that $y(t)$ will approach as $t \to \infty$?

7. For general values of the positive constants k and a, give a discussion of the qualitative behavior of the solutions to the logistic equation (2). Can you describe what the graphs of the solutions look like in the (x, y)-plane?