Math 3CI: Project 10
Geometry of $\mathbb{R}^n$
Revised Version

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1 The dot product

In the last project, we learned how to deal with homogeneous linear systems of equations in many unknowns. It would be helpful if we could utilize some "geometric intuition" in dealing with such systems. But it is hard to draw geometric pictures in say $\mathbb{R}^{10}$, and we often need to deal with systems of equations that involve hundreds of variables. Are there some concepts from geometry, such as angle, that we can make sense of in $\mathbb{R}^n$ for arbitrary n?

If $\mathbf{x} = (x_1 \ldots, x_n)$ and $\mathbf{y} = (y_1 \ldots, y_n)$ are elements of $\mathbb{R}^n$, we define their *dot product* by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

The dot product has several key properties:

1. it is symmetric: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$;
2. it is bilinear: $(a\mathbf{x} + \mathbf{x}') \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) + \mathbf{x}' \cdot \mathbf{y}$;
3. and it is positive-definite: $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

The *length* of an element $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Note that the length of an element $\mathbf{x} \in \mathbb{R}^n$ is always nonnegative.

1.a. If $\mathbf{x} = (1, 3, 2, 5)$ and $\mathbf{y} = (2, 0, 0, 4)$, what is $\mathbf{x} \cdot \mathbf{y}$?

b. If $\mathbf{a} = (1, 1, 1)$ and $\mathbf{x} = (x, y, z)$ where x, y and z are variables, the equation $\mathbf{a} \cdot \mathbf{x} = 0$ describes a plane in $\mathbb{R}^3$. Draw a sketch of that plane.

c. If $\mathbf{a}_1 = (1, 0, -2)$ $\mathbf{a}_2 = (0, 1, -5)$ and $\mathbf{x} = (x, y, z)$, then the set V of solutions to the homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \qquad \mathbf{a}_2 \cdot \mathbf{x} = 0$$

is a linear subspace of $\mathbb{R}^3$. Find a basis for that linear subspace.

d. Draw a sketch in $\mathbb{R}^2$ of the set of vectors of length one:

$$S^1 = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

We would like to be able to discuss the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. The following fact regarding the dot product is helpful:

Cauchy-Schwarz Theorem. *If* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{y} \neq \mathbf{0}$, *then*

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \leq 1. \tag{1}$$

Sketch of proof (included to satisfy your curiosity—you may skip the proof if you want): If $\mathbf{v}$ is any element of $\mathbb{R}^n$, then $\mathbf{v} \cdot \mathbf{v} \geq 0$. Hence

$$(\mathbf{x}(\mathbf{y} \cdot \mathbf{y}) - \mathbf{y}(\mathbf{x} \cdot \mathbf{y})) \cdot (\mathbf{x}(\mathbf{y} \cdot \mathbf{y}) - \mathbf{y}(\mathbf{x} \cdot \mathbf{y})) \geq 0.$$

Expanding using the properties for dot product yields

$$(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})^2 - 2(\mathbf{x} \cdot \mathbf{y})^2(\mathbf{y} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{y})^2(\mathbf{y} \cdot \mathbf{y}) \geq 0$$

or

$$(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})^2 \geq (\mathbf{x} \cdot \mathbf{y})^2(\mathbf{y} \cdot \mathbf{y}).$$

Dividing by $\mathbf{y} \cdot \mathbf{y}$, we obtain

$$|\mathbf{x}|^2|\mathbf{y}|^2 \geq (\mathbf{x} \cdot \mathbf{y})^2 \quad \text{or} \quad \frac{(\mathbf{x} \cdot \mathbf{y})^2}{|\mathbf{x}|^2|\mathbf{y}|^2} \leq 1,$$

and (1) follows by taking the square root.

The key point of the Cauchy-Schwarz Inequality (1) is that it allows us to define angles between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. It follows from properties of the cosine function that given a number $t \in [-1, 1]$, there is a unique angle θ such that

$$\theta \in [0, \pi] \qquad \text{and} \qquad \cos\theta = t.$$

Thus we can define the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ by requiring that

$$\theta \in [0, \pi] \qquad \text{and} \qquad \cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}. \tag{2}$$

Then the dot product satisfies the formula

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\theta,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

2.a. We would like to know whether the above definition agree with the usual definition of angle for vectors in $\mathbb{R}^2$. If $\mathbf{u}$ is a vector of length one in $\mathbb{R}^2$ which makes an angle of α with the x-axis, show that

$$\mathbf{u} = (\cos\alpha, \sin\alpha).$$

b. If $\mathbf{u} = (\cos\alpha, \sin\alpha)$ and $\mathbf{v} = (\cos\beta, \sin\beta)$, what is $\mathbf{u} \cdot \mathbf{v}$?

c. There is a formula from trigonometry which states that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. How does this relate to the result of part b?

d. What is the angle between $\mathbf{x} = (1, 2, 2, 0)$ and $\mathbf{y} = (-1, 1, 1, 1)$?

In particular, we can say that two vectors vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are *perpendicular* or *orthogonal* if the angle between them is 90 degrees, or if $\mathbf{x} \cdot \mathbf{y} = 0$. This provides much intuition for dealing with vectors in $\mathbb{R}^n$. Thus if $\mathbf{a} = (a_1, \dots a_n)$ is a nonzero element of $\mathbb{R}^n$, the homogeneous linear equation

$$a_1x_1 + \cdots + a_nx_n = 0 \tag{3}$$

describes the set of all vectors $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ that are perpendicular to $\mathbf{a}$. We say that (3) is the equation of the *hyperplane* in $\mathbb{R}^n$ which is perpendicular to $\mathbf{a}$.

3.a. What is the angle between the two planes

$$x + z = 0 \quad \text{and} \quad x + 2y + 2z = 0?$$

b. What is the angle between the two hyperplanes

$$x_1 + 2x_2 + 2x_3 = 0 \quad \text{and} \quad -x_1 + x_2 + x_3 + x_4 = 0?$$

Now let us apply these ideas to homogeneous linear system of equations, such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots\cdots & \quad . \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned} \tag{4}$$

The dot product allows us to write such a system in a slightly different form. If we let

$$\begin{aligned} \mathbf{a}_1 &= (a_{11}, a_{12}, \dots, a_{1n}), \\ \mathbf{a}_2 &= (a_{21}, a_{12}, \dots, a_{2n}), \\ . \quad & . \qquad \cdots \\ \mathbf{a}_m &= (a_{m1}, a_{m2}, \dots, a_{mn}), \end{aligned}$$

then the set V of solutions to (4) is just

$$V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_1 \cdot \mathbf{x} = 0,\ \mathbf{a}_2 \cdot \mathbf{x} = 0,\ \cdots,\ \mathbf{a}_m \cdot \mathbf{x} = 0\}. \tag{5}$$

We can say that V is the intersection of m hyperplanes or alternatively, the set of vectors perpendicular to the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

4. Suppose that

$$\begin{aligned} \mathbf{a}_1 &= (2,4,2,4,6), \\ \mathbf{a}_2 &= (1,2,1,3,4), \\ \mathbf{a}_3 &= (3,6,3,6,9). \end{aligned}$$

Find a basis for

$$V = \{\mathbf{x} \in \mathbb{R}^5 : \mathbf{a}_1 \cdot \mathbf{x} = 0,\ \mathbf{a}_2 \cdot \mathbf{x} = 0,\ \mathbf{a}_3 \cdot \mathbf{x} = 0\}.$$

What is the dimension of V?

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are vectors in $\mathbb{R}^n$ and that

$$W = \mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m).$$

We recall from Project 9 that this means that $\mathbf{x} \in V$ if and only if it is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$, that is, if and only if

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_m\mathbf{b}_m,$$

for some choice of real numbers $c_1, \ldots, c_m$. If

$$V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_1 \cdot \mathbf{x} = 0,\ \mathbf{a}_2 \cdot \mathbf{x} = 0,\ \cdots,\ \mathbf{a}_m \cdot \mathbf{x} = 0\},$$

we say that V is the *orthogonal complement* of W and we write $V = W^\perp$.

5. a. Suppose that

$$\begin{aligned} \mathbf{a}_1 &= (1,3,2,4,0,2), \\ \mathbf{a}_2 &= (0,0,1,1,2,1), \\ \mathbf{a}_3 &= (1,3,3,5,2,3), \end{aligned}$$

elements of $\mathbb{R}^6$. Find a basis for

$$W = \mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m).$$

b. Find a basis for the orthogonal complement

$$V = W^\perp = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{a}_1 \cdot \mathbf{x} = 0,\ \mathbf{a}_2 \cdot \mathbf{x} = 0,\ \mathbf{a}_3 \cdot \mathbf{x} = 0\}.$$

2 Matrix multiplication

A useful notation has evolved for writing linear systems of equations, such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \cdots\cdots & \quad . \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{6}$$

an a relatively compact form. Here the a_{ij}'s and b_i's are known elements of $\mathbb{R}$, and we are solving for the unknowns $x_1, \ldots, x_n$. We can write this system in terms of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ b_m \end{pmatrix}$$

as

$$A\mathbf{x} = \mathbf{b}.$$

The matrix product on the left-hand side of this equation is obtained by taking the dot products of the rows of A with $\mathbf{x}$.

6. Suppose that

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}.$$

What is $A\mathbf{x}$?

Matrix products of more general matrices are defined in the same way. Given matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product AB is defined to be the matrix whose entries are the dot products of the **rows** of A with the **columns** of B. Thus if

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \cdot \\ \mathbf{a}_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix},$$

then

$$AB = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_p \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{b}_p \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{b}_p \end{pmatrix}.$$

The number of columns of A must equal the number of rows of B for the matrix product AB to be defined.

7.a. Suppose that

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 5 \\ 7 & 2 \end{pmatrix}.$$

Find AB and BA.

b. Suppose that

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 8 & 5 \end{pmatrix}.$$

Find AB and BA.

c. Suppose that

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 7 \end{pmatrix}.$$

Are AB and BA both defined? Find the products that are defined.

8. (OPTIONAL) This problem uses complex numbers. The *Pauli spin matrices* are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the products

$$\sigma_x\sigma_y, \quad \sigma_y\sigma_x, \quad \sigma_y\sigma_z, \quad \sigma_z\sigma_y, \quad \sigma_z\sigma_x, \quad \sigma_x\sigma_z.$$

9. Suppose that A is a matrix with two rows and two columns. If $\mathbf{x} \in \mathbb{R}^2$, matrix multiplication defines a new vector $A\mathbf{x} \in \mathbb{R}^2$. In this way we obtain a geometric *linear transformation*

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x}$$

from $\mathbb{R}^2$ to itself. In this problem, we want to describe the action of the geometric linear transformation

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As a first step, determine the image of the point

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{that is} \quad A\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then find the image of the point

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{that is} \quad A\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From these facts can you determine what is the image of the point

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}?$$

Can you describe in words what happens to an arbitrary point in $\mathbb{R}^2$ under the linear transformation?

10. a. Use the same technique to describe the action of the geometric linear transformation

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

b. Determine the action of the linear transformation

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

c. Determine the product BA, where A and B are the matrices

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

What does the composition linear transformation

$$\mathbf{x} \mapsto \mathbf{y} = A\mathbf{x} \mapsto \mathbf{z} = BA\mathbf{x}$$

represent? Can you derive a formula for $\cos(\theta + \phi)$?

11. a. Suppose you don't "believe in" complex numbers, but you do accept the rules of matrix multiplication. Your friend says, "Look, I can construct a collection of matrices that behaves just like complex numbers." He then goes on to explain that he can write

$$x + iy \quad \text{for the matrix} \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

He claims that all the usual rules for addition and multiplication of complex numbers hold for such matrices. Is he right?

b. What matrix corresponds to the complex number $e^{i\theta}$? What geometric motion does it correspond to?

c. What matrix corresponds to the complex number $re^{i\theta}$? What geometric motion does it correspond to?