Mathematics 241A, Fall 2009, Topics in Differential Geometry

Professor John Douglas Moore

MWF 1:00-1:50 Girvetz 2127

This survey course will be devoted to global analysis on infinite-dimensional manifolds and its application to the nonlinear partial differential equations that arise in geometry. Although it is only the simplest differential equations, such as the equations for two-dimensional harmonic and minimal surfaces and the Yang-Mills equations on four-manifolds, that can be studied effectively via the theory of infinite-dimensional manifolds, these simplest of nonlinear differential equations have had a profound impact on geometry and topology.

I. We will begin by discussing calculus on manifolds modeled on Banach or Hilbert spaces. Except for the fact that we need to use some theorems from functional analysis (such as the Baire category theorem), calculus on infinite-dimensional manifolds is mostly parallel to the finite-dimensional theory. Main examples of infinite-dimensional manifolds include the manifold $\mathrm{Map}(N, M)$ of continuous maps from N to M, where N and M are finite-dimensional smooth manifolds with N compact. We will describe the basic tools for applications of geometry to nonlinear differential equations, including Sards theorem and de Rham cohomology for infinite-dimensional manifolds.

II. The first application of infinite-dimensional manifolds was to the Morse theory for the ordinary action function J on the manifold $\mathrm{Map}(S^1, M)$ where S^1 is the unit circle, and the manifold M is given a Riemannian metric. A celebrated conjecture of Klingenberg states that a compact Riemannian manifold M with finite fundamental group has infinitely many geometrically distinct smooth closed geodesics. A solution has been found in many cases, using the Morse theory for J. We will describe how de Rham theory on infinite-dimensional manifolds (and Sullivans theory of minimal models) sheds some light on this problem.

III. A second application of global analysis is to harmonic and minimal surfaces in Riemannian manifolds. In this case, one would like to study the Dirichlet energy function E on the space $\mathrm{Map}(\Sigma, M)$, where Σ is a closed Riemann surface. Unfortunately, the usual energy E is not well-behaved when the space of functions is completed with a topology that is strong enough for the techniques of global analysis on infinite-dimensional manifolds, so we have to perturb the energy to get a function E_α which is well-behaved. We can then use Morse theory for the perturbed function E_α to study the perturbed problem and consider the limit as the perturbation parameter is turned off. Using this approach, we will prove the theorem of Sacks and Uhlenbeck that if the compact manifold M is simply connected, generators for $\pi_2(M)$ can be represented by minimal two-spheres. We hope to also discuss the Parker-Wolfson bubble tree and what this says about the possible existence of a partial Morse theory for minimal surfaces.

IV. Finally, assuming there is sufficient time, we will consider the Seiberg-Witten invariants for four-dimensional manifolds. In this case, the infinite-dimensional manifold of interest is the space of gauge equivalence classes of connections on a line bundle over the four-dimensional manifold M, and we can apply global analysis techniques to the Seiberg-Witten equations, coupled equations consisting of the Dirac equation and a slightly nonlinear equation for the curvature of a connection in a line bundle over M. The Seiberg-Witten equations show that certain four-dimensional manifolds have more than one smooth structure, and that other smooth four-manifolds have no smooth structures at all. It is remarkable that the smooth structure on a four-dimensional manifold depends on the "moduli space" of solutions to a certain nonlinear system of partial differential equations associated to the manifold. This illustrates that on a very fundamental level, geometry, topology and nonlinear partial differential equations are closely related.