The length scale of 3-space knots, ephemeral knots, and slipknots in random walks

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The probability that a random walk or polygon in the 3-space or in the simple cubic lattice contains a small knot, an ephemeral knot, or a slipknot goes to one as the length goes to infinity. The probability that a polygon or walk contains a "global" knot also goes to one as the length goes to infinity. What immerges is a highly complex picture of the length scale of knotting in polygons and walks. Here we study the average scale of knots, ephemeral knots, and slipknots in 3-space random walks, paying special attention to the probability of their occurance and to the growth of their average sizes as a function of the length of the walk.

§1. Introduction

Self-avoiding random walks and polygons in 3-space or in the cubic lattice provide a popular model for linear polymers under certain physical conditions. With increasing length, the probability that a random walk or polygon contains a knot goes to one[1),2),4),5)] proving a conjecture of Frisch and Wasserman[6)] and of Delbruck.[7)] In addition, the probability that a random walk or polygon contains a knotted portion of any given topological type goes to one as the length also goes to infinity. As a consequence, the influence of knotting on the biological and physical properties of polymers,[8),9)] their occurrence in DNA[10)] and in proteins,[11)–13)] and their impact upon the scaling of of physical properties of macromolecules[14)–17)] have inspired wide ranging theoretical and experimental research. In this paper, we study the length scaling of the knotted portion of a random walk and extend this to encompass more fragile knotted structures, that of an ephemeral knot and its associated slipknot. These occur in both random walks or polygons and have recently been found in biological structures. An *ephemeral knot* in a random walk or polygon is a knotted portion that is contained in a larger unknotted portion which we will call a *slipknot*. Slipknots have been identified in proteins.[18)] They also occur in random walks and 3-space and in the simple cubic lattice with a probability that also goes to one as the length goes to infinity.[19)] As a consequence, it seems reasonable to conjecture that they are also present in a wide range of other polymers. They appear, however, to have avoided experimental or numerical detection to date.

For polymers at θ condition, one expects that knots are quite localized. The specific nature of the localization and the measures of the relative scale are of great interest.[14),20)–23)] The knotted portions of the random walk or polygonal chain used to model the structure of the polymers have an impact upon the spatial conformation of the chain and, as a consequence, correlate with the physical properties of the modeled polymers. Because of the similarity of the spatial conformations, we propose that slipknots will have an analogous impact upon the associated physical properties of polymers. To distinguish the nature of the knotting, *global knotting* occurs when the average support of the knot, i.e. the smallest subsegment of the walk or polygon, is proportional to the length of the walk or polygon. If the scaling of the average support is proportional to N^{λ}, for $0 < \lambda < 1$, we say that the knotting is *weakly local*. The knotting is *strongly local* if the support is of bounded length as the length of the walk or polygon goes to infinity. Marcone et. al.[21)–23)] observe that, for random polygons in the simple cubic lattice, the length of the knotted portion of a circular chain scales as $N^{3/4}$. What makes this observation all the more important is that the original proofs of the asymptotic presence of knots actually show that one could add the provision of bounded length and arrive at the same conclusion proving the existence of a substantial presence of strongly local knots. Diao et al[24)] have proved that global knots are present with probably approaching one as the length goes to infinity as well. Thus, the influence of excluded volume considerations on the scale of knotting may be critical so that the scaling of the average length is uncertain in both random walks or polygons.

We will observe that the average support of knots, ephemeral knots, and slipknots in random walks in 3-space are strongly local in nature, i.e. their average lengths are bounded. The influence of excluded volume plus the closure required to produce random polygons appears to account for the difference between the scaling we observe for random walks and the weakly local average support of knots in random walks that one would expect based on the research on the simple cubic lattice.

We first describe our method for the detection of the knotting of segments of a 3-space random walks.[19),25),26)] Using these criteria, we give an extension of the methods for knots to study the probability that a random walk contains an ephemeral knot, and its companion slipknot, goes to one as the length goes to infinity. We next describe how we use the detection methods to identify the presence of knots, ephemeral knots and slipknots in random walks and to identify its support.

§2. Knots, ephemeral knots, and slipknots

A closed polygon in 3-space is knotted if there is no ambient deformation of Euclidean space taking the polygon to the standard planar circle. The search for computationally efficient and effective methods to determine the specific structure of knotting for polygons in 3-space is a continuing challenge for topology. More importantly for this research, the search for an appropriate formulation of knotting of open polygons is even less well understood. From a classical topological point of view, knotting of open polygons is not topologically possible because, if edge lengths are allowed to vary, each open polygon is ambient isotopic to a standard interval in the “x”-axis in 3-space (This argument is popularly called the “light-bulb” theorem) and, as a consequence, open polygons are topologically unknotted. They may, however, be geometrically knotted when the edge lengths are fixed. This is confirmed by the examples of Canterella-Johnson and others.[27)–29)] If one considers equilateral polygons, whether or not there are configurations that cannot be deformed to a straight segment preserving the edge lengths is unknown. The analogous problem for closed equilateral polygons is also unknown, i.e. "Are there topologically unknotted equilateral polygon configurations in 3-space that cannot be deformed to the standard planar equilateral polygon?" These two questions give a sense of the degree of difficulty in describing the knot theory of equilateral polygons in 3-space beyond those having 8 or fewer edges for which one knows more.[30)–32)]

In order to study the knotting of open off-lattice, or 3-space, polygons and being concerned with the features of existing methods, Millett, Dobay and Stasiak[25),26)] proposed a statistical method to identify and describe, quantitavely, the nature of the knotting present in an open polygon. This method was evaluated in a study of knotting in random walks and tested against the previously identified knotting present in protein structures. The *MDS Method* can be described as follows: given an open polygonal arc, consider the probability distribution, or *knotting spectrum*, arising from connecting both endpoints of the polygon to points on the sphere "at infinity," as the points on the sphere vary with respect to the uniform distribution. For all practical purposes, this spectrum identifies a dominant knot type at the 0.50 level, meaning that a single knot type occurs in more than half the closures. In a test of one thousand 300 step random walks in 3-space, the 0.50 level test was successful 99.6% of the time.[25)] As a consequence, it appears that the MDS approach provides a powerful method with which to analyze the knotting of open chains. It is the principal criterion employed in this paper.

2.1. *The asymptotic presence of knots, ephemeral knots, and slipknots*

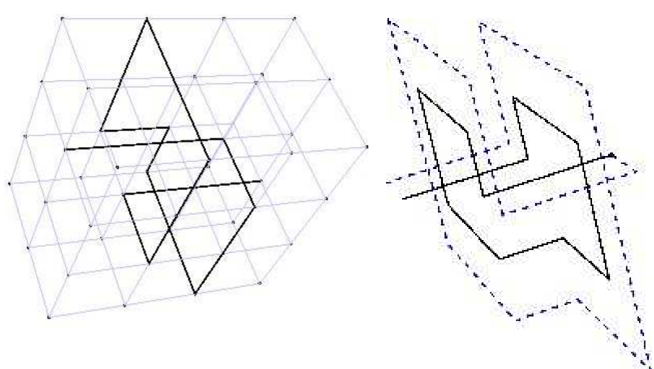

Fig. 1. Sumners-Whittington trefoil knot and Millett trefoil slipknot in the integer lattice

The key theorems are:

Theorem 2.1. Sumners and Whittington, Pippenger[1),2)] *All except exponentially few sufficiently long self-avoiding polygons on the simple cubic lattice contain a knot of any given topological type.*

The key to the proof is the application of the Kesten Pattern Lemma[3)] to the knotted trefoil pattern in Figure 1. The same method can be applied to the slipknotted pattern shown in Figure 1. Observe that this configuration can not be "reknotted" for the same reason that the Sumners-Whittington trefoil can not be unknotted, it is impossible for any other portion of the walk or polygon to penetrate the convex hull of the configuration. Thus we have a tight trefoil slipknot realized by a Kesten Pattern.

Theorem 2.2. *(Millett[19]) All except exponentially few sufficiently long self-avoiding walks or polygons on the simple cubic lattice contain ephemeral knots of any fixed topological type and their associated slipknots.*

Immediate consequences of this theorem include the following:

Corollary 2.3. *The probability that a self-avoiding walk or polygon on the simple cubic lattice contains an ephemeral knot and its associated slipknot (of any selected topological type) goes to one as the length goes to infinity.*

Corollary 2.4. *(Whittington) The probability that a knotted self-avoiding walk or polygon in the simple cubic lattice contains a slipknot goes to one as the length goes to infinity.*

Applying the MDS method to the trefoil slipknot in Figure 1, for a sample of size 10000, one has a knotting spectrum consisting of 86.7% unknots and 13.3% trefoils. If one applies the MDS method to the ephemeral trefoil subsegment, one has a knotting spectrum consisting of 6.3% unknots, 92.76% trefoil knots, 0.44% figure-eight knots and, 0.47% 5_1 knots thereby confirming its identification.

Results directly analogous to those we have just described for the simple cubic lattice are also true in 3-dimensional Euclidean space. The keystone is the theorem of Yuanan Diao[5] whose proof can be extended to prove:

Theorem 2.5. *(Millett[19]) All except exponentially few sufficiently long self-avoiding random walks and polygons in* 3*-space contain ephemeral knots of any fixed topological type and their associated slipknots.*

Corollary 2.6. *The probability that self-avoiding polygon or random walk in* 3*-space contains an ephemeral knot of any fixed topological type and its associated slipknot goes to one as the length goes to infinity.*

§3. MDS Identification of knots, ephemeral knots, and slipknots

The power of the MDS method[25,26] in identifying knots, ephemeral knots, and knots in chains and polygons is a consequence of several fundamental properties that one might desire in a computationally stable method. The first concerns that numerical characteristics of the method.

`Robust`: A definition of knotting of an open chain is *robust* if small perturbations of the vertices in a generic chain do not change the knot type.

The second concerns its relationship with classical notions of knotting in closed polygons.

`Continuity`: A definition of knotting of an open chain is *continuous* if when it is applied to a generic small break in an edge of a closed polygon, one identifies the same knot type as that of the closed polygon.

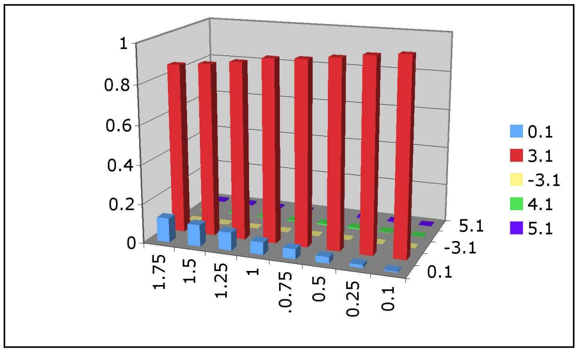

Fig. 2. The knotting spectrum evolution as the gap size gets smaller

As an example of the continuous evolution the open chain, we consider a closed trefoil from which one has removed an edge. Growing the missing edge from one end we see, as the size of gap between the ends is closed so that the gap approaches zero, that the spectrum evolves to that of the trefoil as shown in Figure 2.

Theorem 3.1. *The MDS definition of knotting is robust and continuous.*

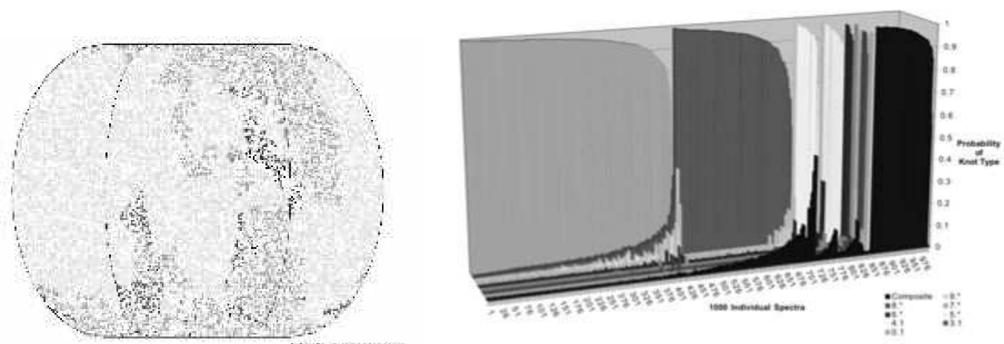

Fig. 3. Spherical distribution of knot types and the knotting spectra of 1000 random walks

We should note that there are a relatively small number of open equilateral polygons for which no single knot type is selected at the 50% level. The dependence of the not type on the choice of spherical closure is show in Figure 3. In a sample of 1000 random walks of 300 steps, we found only four, i.e. 0.4%, not having an identified knot type. Their spectra are shown in Figure 3.

Thus, to determine whether a random walk is knotted or not, we apply the MDS method to the entire walk. To search for ephemeral knots in an unknotted random walk or the size of the knot in a knotted random walk, we analyze all segments with increasing length starting at length four (we know that at least six edges are required for knotting). Given an ephemeral knot, i.e. a knotted segment in an unknotted random walk, the slipknot containing it is identified by searching, in increasing length, the subsegments of the walk containing the ephemeral knot.

§4. Global, weakly local, and strongly local knotting

Given a measure of the support of a knotted segment in an open or in a closed polygon of length n, how does one distinguish between strongly local, weakly local, and global knotting phenomena? At one extreme, for a fixed knot type, if the average length of the knotted segments of this knot type is proportional to the length of the polygon, one would have an example of *global knotting* even though such knots may engage only a small, but fixed, fraction of the polygon. The critical factor is that the length of knotting scales linearly with the length of the polygon. Note, however, for each knot type there is a minimal number of edges or steps that is required to exhibit the knot type so that this criterion may only be relevant for polygons with a number of edges much larger when compared to these lower bounds. Another way of thinking of this is that, for global knotting, the ratio of the average length supporting the knot type to the length of the polygon is bounded away from zero as the length goes to infinity.

Marcone et. al.[21),22)] have described knots whose average length grows in proportion to a positive power, less than unity, of the length. These are *weakly locally knotted*. They found an exponent of $\frac{3}{4}$ thereby implying the the fraction of the polygon implicated, on average, in knotting to be proportional to $n^{\frac{-1}{4}}$ and giving an explicit measure of the rate at which the ratio tends to zero.

Strong local knotting occurs, for a specific knot type, when the average length of the knotted portion is bounded as the length goes to infinity and, therefore, the proportion implicated in knotting of this topological type goes to zero at a rate proportional to n^{-1}.

Diao et. al.[24)] describe another approach to distinguishing between local and global knotting. They show that equilateral random polygons have the property of global knottedness. Jungreis[33)] showed earlier that Gaussian random polygons had the property of global knottedness. It is possible that a model has the property of global knottedness even though the probability of the presence of local knots goes to one with increasing length.

§5. Simulation results

Off lattice random walks are generated by selecting unit vectors with respect to the uniform distribution. In order to give an estimation of the MDS knotting spectrum of an open polygonal knot in 3-space, including those in the simple cubic lattice, we first determine smallest spherical ball containing the configuration using the miniball algorithm.[34)] As a surrogate for the sphere at infinity centered we take 250 points, with respect to the uniform distribution, on the sphere centered at this point and whose radius is 100 times that of radius of the smallest containing sphere and form the close of the ends of the open polygon to each of these points. For each closed polygon thus formed we compute its Ewing-Millett[35)] code from a projection to the ‘XY”-plane and use their program to compute the HOMFLY polynomial[?] as a representation of the knot type. While there are topologically distinct knots with identical HOMFLY polynomials, they do provide a highly effective discriminant in this range.[36] The HOMFLY polynomial histogram for the 250 closures provides the knotting spectrum for the open polygon.

For each random walk, the MDS method is applied to determine the knot type of the walk. If the walk is unknotted, segments are examined with increasing length to identify whether or not the segment is knotted. Once a knotted segment is encountered it is identified as an ephemeral knot and the smallest unknotted segment containing it is identified, giving the associated slipknot. If the walk is knotted, its knot type is recorded and segments are examined, with increasing length, until the shortest segment having the knot type of the walk is identified.

5.1. *The random walk data and its implications*

The length distributions of knots and ephemeral knots in random walks give a clear expression of the strongly local nature of knotting in random walks of 500 steps shown in Figure 4. The knots have mean length of 19 with a standard deviation of 23 while the ephemeral knots have a mean length of 41 and a standard deviation of 59.

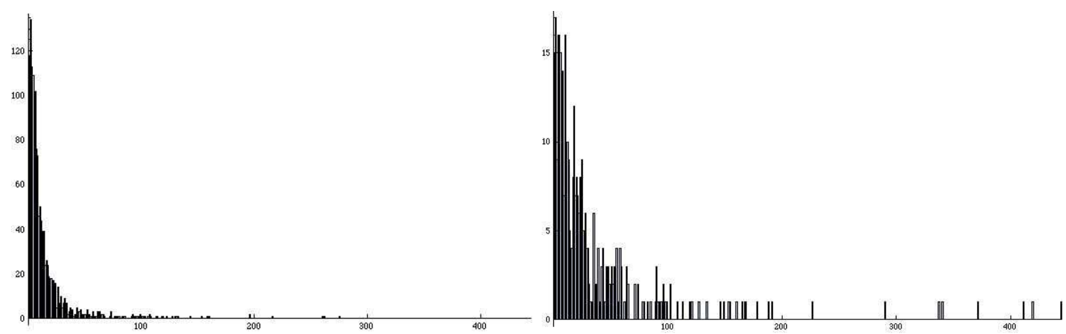

Fig. 4. The length distributions of knots and ephemeral knots in 500 step random walks

Table I shows the growth in frequency and in length of the knots, ephemeral knots and, slipknots in random walks as the length of the walk increases. Already at walks of 150 steps, more than 40% contain a knot or an ephemeral knot, see Figure 5. The growth of the average lengths of knots, ephemeral knots, and slipknots is shown in Figure 5 including a comparison with the $n^{3/4}$ length growth observed for simple cubic lattice polygons identified by Marcone, et. al. [22] The probability of local knotting goes to one as the length of the walk goes to infinity[1,2,4,5] and, as we have observed, the same is true for ephemeral and slipknots. The data reported here using the MDS strategy to identify the knotted segments in the random walk shows the strong localization of knotting in random walks.

§6. Conclusions

King et. al.[18] report the the identification of a ephemeral knot, i.e. a slipknot, deeply embedded in the structure of *Escherichia coli* alkaline phosphatase (PDB code 1ALK[37]). They find that the smallest segment of the protein that is knotted

Table I. Knotting in Random Walks

Walk	*% Knotted*	*Length*	*% Ephemeral*	*Ephemeral Length*	*Slipknot Length*
25	2.00 %	11.130	0.98 %	13.551	15.122
50	6.70 %	15.876	4.53 %	19.537	21.323
75	13.75 %	18.320	7.85 %	24.255	26.008
100	19.50 %	20.623	12.05 %	27.000	30.119
150	30.40 %	22.321	17.15 %	28.133	37.729
200	39.35 %	24.712	20.70 %	34.944	37.907
250	48.87 %	23.732	22.19 %	36.269	39.380
300	52.85 %	25.159	21.59 %	40.169	45.928
350	64.90 %	21.972	21.30 %	42.149	46.732
400	62.59 %	21.777	20.02 %	43.066	49.578
450	78.23 %	20.623	19.90 %	41.735	48.403
500	76.31 %	19.036	19.23 %	41.193	53.308

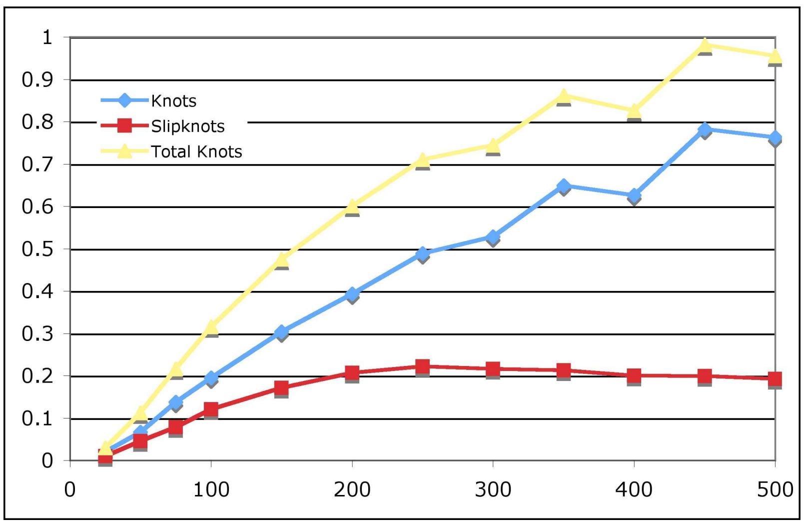

Fig. 5. Proportion of Knots and Ephemeral Knots

is between residues 51 and 371, forming a right-handed trefoil. As approximately 50 residues precede the segment on the N-terminal side and, on the C-terminal end, residues 371 to 419 forming a loop that eventually ‘unknots” the trefoil. They found three other distinct protein folds, with PDB codes 1P6X, 2NWL, and 2A65, having deeply embedded ephemeral knots as well as an additional candidate, 2J85, which was not as deeply embedded as the others. They undertook an intense analysis of the structure of each of these proteins. Here, we have proved that such structures occur in random walks and random equilateral polygons with asymptotic probability one. Furthermore, we report simulation data that supports the conjecture that the average structure knots, ephemeral knots, and slipknots is strongly local with increasing length, i.e. that the average lengths of these structures is bounded. In addition, we see that the presence of these various forms of knotting grows so rapidly that more than 90% of the lengths longer than 350 steps are likely to contain knotted

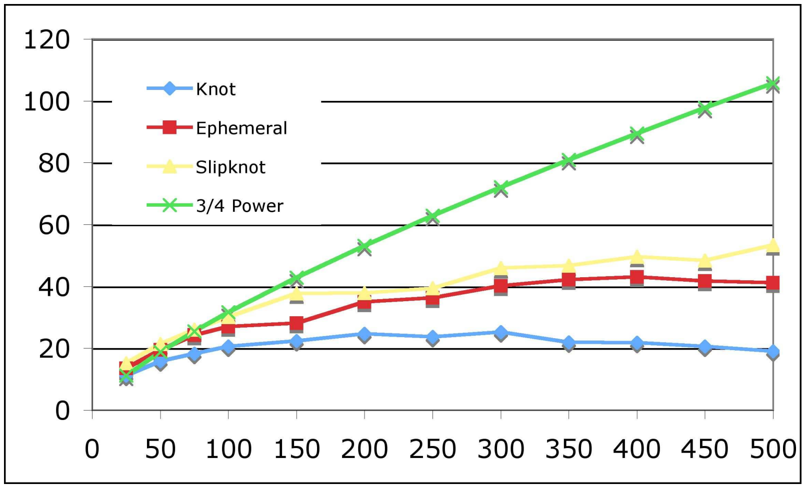

Fig. 6. Lengths of Knots, Ephemeral Knots and, Slipknots

regions.

What influence does the strong localization of knots, ephemeral knots, and slipknots have on the biological or physical properties of a polymer? We have provided one perspective on this by showing the impact on the radius of gyration. Specifically, the strong localization causes the unknotted lengths to have a disproportionate contribution to the radius of gyration in comparison to the knotted length. We conjecture that the importance of this disproportion may continue to increase with increasing length.

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