MAT 145 : Odds and Ends

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Example 5.8 in Crossley's Book

This example establishes a particular homeomorphism between the closed unit disk D^2 and the square $[-1,1]^2$, both considered as subspaces of $\mathbb{R}^2$. Recall the definitions

$$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
$$[-1,1]^2 = \{(x,y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\} \ .$$

The functions f and g are well-defined

One begins by defining a function $f : D^2 \to [-1,1]^2$ by $f(0,0) = (0,0)$, and

$$f(x,y) = \left(\frac{\sqrt{x^2+y^2}}{\max\{|x|,|y|\}} \right) (x,y) \quad \text{otherwise} \ .$$

We actually need to check that the co-domain for f is correct, that is, $f(x,y) \in [-1,1]^2$ for every $(x,y) \in D^2$; this it is not so obvious from the formula for $f(x,y)$. Let $(x,y) \in D^2$ be given. Let $r = \sqrt{x^2+y^2}$; so $0 \leq r \leq 1$. Furthermore, assume that $(x,y) \neq (0,0)$; so $r \neq 0$. Now, $\max\{|x|,|y|\} = |x|$ if and only if $|x| \geq |y|$. Suppose that this is the case. Then

$$\begin{aligned} f(x,y) &= \frac{r}{|x|}(x,y) \\ &= \left(\frac{rx}{|x|}, \frac{ry}{|x|} \right) \ . \end{aligned}$$

So both coordinates of $f(x,y)$ are in the interval $[-r,r] \subset [-1,1]$; keep in mind that we are assuming $|x| \geq| y|$, so $\frac{ry}{|x|} \in [-r,r]$. Therefore $f(x,y) \in [-r,r]^2 \subset [-1,1]^2$. In the case that $|x| \leq| y|$, a similar calculation that shows that $f(x,y) \in [-r,r]^2 \subset [-1,1]^2$. In fact, we showed that $f(\mathbb{S}^1_r) \subset C_r$ where $\mathbb{S}^1_r = \partial\big(B_r(0,0)\big)$ and $C_r = \partial\big([-r,r]^2\big)$, $0 < r \leq 1$. A similar calculation shows that the function $g : [-1,1] \to D^2$, defined by $g(0,0) = (0,0)$ and

$$g(x,y) = \left(\frac{\max\{|x|,|y|\}}{\sqrt{x^2+y^2}}\right)(x,y) \ \text{ otherwise },$$

satisfies $g(C_r) \subset \mathbb{S}^1_r$ for every $r \in (0,1]$. It follows that g has the correct co-domain.

The functions f and g are inverses of each other

We show that $(g \circ f)(x,y) = (x,y)$ for all $(x,y) \in D^2$ with $|x| \geq |y|$; the calculation for $|x| \leq| y|$ is similar.

$$\begin{aligned}
(g \circ f)(x,y) &= g\left(\frac{rx}{|x|}, \frac{ry}{|x|}\right) \\
&= \left(\frac{\left|\frac{rx}{|x|}\right|}{\sqrt{\left(\frac{rx}{|x|}\right)^2 + \left(\frac{ry}{|x|}\right)^2}}\right)\left(\frac{rx}{|x|}, \frac{ry}{|x|}\right) \\
&= \left(\frac{r}{\sqrt{\frac{r^4}{|x|^2}}}\right)\left(\frac{rx}{|x|}, \frac{ry}{|x|}\right) \\
&= \frac{|x|}{r}\left(\frac{rx}{|x|}, \frac{ry}{|x|}\right) \\
&= (x,y) \ .
\end{aligned}$$

Then a similar calculation shows that $(f \circ g)(x,y) = (x,y)$ for all $(x,y) \in [-1,1]^2$. Therefore f and g are inverses of each other. Therefore, we can promote the mere containment $f(\mathbb{S}^1_r) \subset C_r$ to the *equality* $f(\mathbb{S}^1_r) = C_r$ for all $0 < r \leq 1$. We deduce that $f(\overline{B_r(0,0)}) = [-r,r]^2$ for every $0 < r \leq 1$.

The functions f and g are continuous

We first show that f is continuous at $(0,0)$. Let $\epsilon > 0$ be given; we show that there is a $\delta > 0$ for which $f(B_\delta(0,0)) \subset B_\epsilon(0,0)$. We set $\delta = \frac{\epsilon}{\sqrt{2}}$. Then $f(\overline{B_\delta(0,0)}) = [-\delta,\delta]^2 \subset \overline{B_\epsilon(0,0)}$; indeed, C_δ is circumscribed by $\mathbb{S}^1_\epsilon$ (draw a picture in $\mathbb{R}^2$).

We now finish the proof that f is continuous. Let X be the union of the lines $\{y = x\}$ and $\{y = -x\}$ in $\mathbb{R}^2$. Define $A = \{(x,y) \in D^2 : |x| \geq |y|\}$ and $B = \{(x,y) \in D^2 : |x| \leq |y|\}$; note that $A \cap B = \partial A = \partial B = X \cap D^2$ (draw these regions in $\mathbb{R}^2$). For any $(x,y) \in D^2$, it is easy to see that

$$\max\{|x|,|y|\} = |x| \iff (x,y) \in A \ , \ \text{and}$$
$$\max\{|x|,|y|\} = |y| \iff (x,y) \in B \ .$$

The restriction of the function $f(x,y)$ to A has the formula

$$f(x,y) = \left(\frac{\sqrt{x^2+y^2}}{|x|}\right)(x,y), \ \text{for} \ \ (x,y) \neq (0,0) \ .$$

The restriction of the function $f(x,y)$ to B has the formula

$$f(x,y) = \left(\frac{\sqrt{x^2+y^2}}{|y|}\right)(x,y), \ \text{for} \ \ (x,y) \neq (0,0) \ .$$

We explain why the restriction of $f(x,y)$ to A is continuous; the reasoning is similar for B. A consists of two regions

$$A_+ = A \cap \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \ , \ \text{and}$$
$$A_- = A \cap \{(x,y) \in \mathbb{R}^2 : x \leq 0\}$$

with $A_+ \cap A_- = (0,0)$. It is not difficult to see that A_+ and A_- are closed in A. In A_+, the formula for $f(x,y)$ is simply

$$f(x,y) = \left(\frac{\sqrt{x^2+y^2}}{x}\right)(x,y), \ \text{for} \ \ (x,y) \neq (0,0) \ .$$

The scaler function $(x,y) \mapsto \frac{\sqrt{x^2+y^2}}{x}$ is continuous, and scaler multiplication is continuous (as you can check). So the restriction of f to A_+ is continuous, Similarly, the restriction of f to A_- is continuous. Since f is also continuous at $(0,0)$, we conclude that f is continuous on all of A. Similarly, f is continuous on B.

The sets A and B are closed in $\mathbb{R}^2$ (hence closed in D^2), so f is continuous on all of D^2 (by the Pasting Lemma). The function g is continuous by a similar argument. It follows that f is a homeomorphism.

Metrizability of $\mathbb{R}^{\infty}_{product}$ and Nonmetrizability of $\mathbb{R}^{\infty}_{box}$

The material in this section is taken from the book "Topology: A first course" by James R. Munkres; in that book, the set of infinite sequences in $\mathbb{R}$ is denoted by $\mathbb{R}^{\omega}$. In these notes, the set of infinite sequences in $\mathbb{R}$ is denoted by $\mathbb{R}^{\infty}$.

Theorem (Metrizability of $\mathbb{R}^{\infty}_{product}$)**.** *There exists a metric on $\mathbb{R}^{\infty}$ whose metric topology is the same as $\mathbb{R}^{\infty}_{product}$. Hence $\mathbb{R}^{\infty}_{product}$ is metrizable.*

Proof. First, define a bounded metric on $\mathbb{R}$. Define $\bar{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{d}(a, b) = \min\{|a - b|, 1\} .$$

To check that $\bar{d}$ is a metric on $\mathbb{R}$, it suffices to check the triangle inequality; the other criteria for $\bar{d}$ to be a metric are trivial to check. Let $a, b, c \in \mathbb{R}$. By the triangle inequality of the usual metric on $\mathbb{R}$, we have that

$$|a - c| \leq | a - b| + |b - c| .$$

There are two cases:

1. $\bar{d}(a, b) = |a - b|$ and $\bar{d}(b, c) = |b - c|$.
2. $\bar{d}(a, b) = 1$ or $\bar{d}(b, c) = 1$.

In the first case,

$$\begin{aligned} \bar{d}(a, c) &\leq |a - c| \\ &\leq |a - b| + |b - c| \\ &= \bar{d}(a, b) + \bar{d}(b, c). \end{aligned}$$

In the second case, say, $\bar{d}(a, b) = 1$,

$$\begin{aligned} \bar{d}(a, c) &\leq 1 \\ &= \bar{d}(a, b) \\ &\leq \bar{d}(a, b) + \bar{d}(b, c). \end{aligned}$$

If $\bar{d}(b, c) = 1$, we arrive at the same conclusion. Therefore, $\bar{d}$ is a metric on $\mathbb{R}$.

Now define $D : \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \to \mathbb{R}$ by

$$D(x, y) = \text{lub}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}_{i \in \mathbb{N}}$$

where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$; $D(x, y)$ is defined because $\frac{\bar{d}(x_i, y_i)}{i}$ is bounded above by 1, for all $i \in \mathbb{N}$.

To show that D is a metric on $\mathbb{R}^\infty$, it suffices to establish the triangle inequality; the other criteria for D to be a metric are trivial to check. Let $x, y, z \in \mathbb{R}^\infty$. Let $i \in \mathbb{N}$ be given. Since $\bar{d}$ is a metric on $\mathbb{R}$, we have

$$\bar{d}(x_i, y_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i) \ .$$

Dividing by i yields

$$\begin{aligned} \frac{\bar{d}(x_i, z_i)}{i} &\leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \\ &\leq D(x, y) + D(y, z) \ . \end{aligned}$$

Therefore, $D(x, z) \leq D(x, y) + D(y, z)$. This establishes that D is a metric on $\mathbb{R}^\infty$.

Let $\mathcal{T}_P$ denote the product topology on $\mathbb{R}^\infty$, and let $\mathcal{T}_D$ denote the metric topology induced by D. We show that $\mathcal{T}_P = \mathcal{T}_D$.

The first thing to show is that $\mathcal{T}_D \subset \mathcal{T}_P$. It suffices to show that every metric ball lies in $\mathcal{T}_P$. Let $z \in \mathbb{R}^\infty$ and let $r > 0$; our strategy will be to show that every point of $B_r(z)$ is an interior point. Let $x \in B_r(z)$. Since $B_r(z) \in T_D$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset B_r(z)$ [1]. There exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Now set

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots \ .$$

It is easy to see that $V \in \mathcal{T}_P$ is a neighborhood of x. We now assert that $V \subset B_\epsilon(x)$. Let $y \in V$. Then for all $i > N$, we have

$$\bar{d}(x_i, y_i) \leq 1 \Longrightarrow \frac{\bar{d}(x_i, y_i)}{i} < \frac{1}{N} \ .$$

Since $y \in V$, we have $\frac{\bar{d}(x_i, y_i)}{i} < \frac{\epsilon}{i} \leq \epsilon$ for all $i = 1, \dots, N$; we also have $\frac{\bar{d}(x_i, y_i)}{i} < \frac{1}{N} < \epsilon$ for all $i > N$. So $D(x, y) < \epsilon$; hence $y \in B_\epsilon(x)$. This establishes that $V \subset B_\epsilon(x)$. Clearly $x \in V$, so x has a neighborhood $V \in \mathcal{T}_P$ such that $x \in V \subset B_r(z)$. Therefore, $B_r(z) \in \mathcal{T}_P$.

We now show that $\mathcal{T}_P \subset \mathcal{T}_D$. It suffices to prove that every basis element of $\mathcal{T}_P$ lies in $\mathcal{T}_D$. So let $V = V_1 \times V_2 \times \cdots$ be a basis element of $\mathcal{T}_P$; so $V_i = \mathbb{R}$ for every i except for finitely values $i \in \{\alpha_1, \dots, \alpha_N\} \subset \mathbb{N}$. Let $x \in V$. For each $i = 1, \dots, N$, choose $\epsilon_{\alpha_i} > 0$ small enough so that $\epsilon_{\alpha_i} < 1$ and $(x_{\alpha_i} - \epsilon_{\alpha_i}, x_{\alpha_i} + \epsilon_{\alpha_i}) \subset V_{\alpha_i}$. Set $\epsilon = \min\left\{\frac{\epsilon_{\alpha_i}}{\alpha_i} : i = 1, \dots, N\right\}$. We

[1] It suffices to set $\epsilon = r - D(x, z)$.

now assert that $B_\epsilon(x) \subset V$. Let $y \in B_\epsilon(x)$. Then $\frac{\bar{d}(x_{\alpha_i}, y_{\alpha_i})}{\alpha_i} < \epsilon$ for all $i = 1, \dots, N$. Thus $\bar{d}(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_{\alpha_i} < 1$ for all $i = 1, \dots, N$; hence $|x_{\alpha_i} - y_{\alpha_i}| < \epsilon_{\alpha_i}$ for all $i = 1, \dots, N$. It now follows that $y \in V$; thus $B_\epsilon(x) \subset V$. Therefore $V \in \mathcal{T}_D$.

This establishes that $\mathcal{T}_P = \mathcal{T}_D$. $\square$

The story for $\mathbb{R}^\infty_{box}$ is different.

Theorem (Nonmetrizability of $\mathbb{R}^\infty_{box}$)**.** *There exists no metric on $\mathbb{R}^\infty$ whose metric topology gives $\mathbb{R}^\infty_{box}$. Hence $\mathbb{R}^\infty_{box}$ is not metrizable.*

Proof. The strategy here is to show that the Sequence Lemma[2] does not hold. Define the subspace

$$A = \{(x_1, x_2, \dots) : x_i > 0 \text{ for all } i \in \mathbb{N}\} \subset \mathbb{R}^\infty_{box} .$$

Let $x \in \mathbb{R}^\infty_{box}$ denote the zero-element, that is, $x = (0, 0, \dots)$.

First, note that $x \in \overline{A}$: any basis-element neighborhood

$$V = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

of x must satisfy $a_i < 0 < b_i$ for all $i \in \mathbb{N}$; thus $(\frac{b_1}{2}, \frac{b_2}{2}, \dots) \in V \cap A$.

Now we show that no sequence $\{x_n\} \subset A$ converges to x; in fact, we will show that x has a neighborhood V' so that $x_n \notin V'$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\} \subset A$ is a sequence. For each $n \in \mathbb{N}$, we write x_n as $x_n = (b_{n,1}, b_{n,2}, \dots)$. Now set

$$V' = (-b_{1,1}, b_{1,1}) \times (-b_{2,2}, b_{2,2}) \times \cdots .$$

It is clear that V' is a neighborhood of x. Since $b_{n,n} \notin (-b_{n,n}, b_{n,n})$, it follows that $x_n \notin V'$ for any $n \in \mathbb{N}$. So the sequence $\{x_n\}$ does not converge to x. Therefore, the Sequence Lemma does not hold in $\mathbb{R}^\infty_{box}$. We conclude that $\mathbb{R}^\infty_{box}$ is not metrizable. $\square$

[2] Recall that the Sequence Lemma asserts that if X is a metric space and $A \subset X$ is subset with $x \in \overline{A}$, then there exists a sequence $\{x_n\} \subset A$ for which $x_n \to x$.