MAT 145 : Midterm Exam Study Guide

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Remarks

The exam will consists of proofs. Your exam solutions are expected to be written clearly. You may apply theorems (without proof) if they were proved in class or in Crossley's book, Hatcher's notes, or the set theory notes. In particular, you may not use results proved on homework: it is possible that an exam question may actually be (part of) a homework problem. If you are not sure about whether or not you can apply a result while taking the test, you may quietly ask me.

Practice Problems: All of the *problems for understanding* from Hw 1, 2, and 3 (except Problem 15 in Chapter 1 of Hatcher's notes), and all problems from Quizzes 1 and 2 (including Problem 3 of Quiz 2).

Set Theory (Basics)

Definitions: **Sets**: elements, subsets, index set, union, intersection, complement, difference of sets, Cartesian product, element-chasing; **Functions**: domain (source), co-domain (target), image, pre-image (inverse image), injection, surjection, bijection, composition, inverse function, identity function.

Major Facts/Results: Distributive Laws and DeMorgan's Laws.

Topological Spaces and Continuous Functions (Basics)

Definitions: Topology (collection of "open sets") on a set (in terms of the axioms), "closed sets" with respect to a topology (and how to rephrase the axioms for a topology in terms of closed sets), indiscrete topology for any set X, discrete topology for any set X, topological space, neighborhood of a point in a topological space, continuous function from a topological space to another.

Major Facts/Results: If $f : X \to Y$ is a function from a space X to another space Y, then f is continuous if and only if the pre-image of closed sets are closed. Compositions of continuous functions are continuous.

Major Examples: Topologies on finite sets, usual topology on $\mathbb{R}^n$ (in terms of open balls of the form $B_r(a)$ where $a \in \mathbb{R}^n$ and $r > 0$), lower limit (also called half-open interval) topology on $\mathbb{R}$ (written as $\mathbb{R}_l$ or $\mathbb{R}_h$).

Subspace Topology (Basics)

Definitions: Subspace topology.

Major Facts/Results: The restriction of a continuous function to a subspace is continuous. If X is a space and Y is an open (resp. closed) subspace, then open (resp. closed) sets in Y are also considered open (resp. closed) in X.

Major Examples: Unit sphere $\mathbb{S}^n$ as a subspace of $\mathbb{R}^{n+1}$ (for $n = 0, 1, 2$), torus, cylinder (annulus), Möbius band.

Basis for a Topology

Definitions: Basis for a given topology, basis element, topology generated by a basis.

Major Facts/Results: Criterion for a collection $\mathscr{B} = \{B_\alpha\}$ of subsets of X to form a basis for a topology on X (see Proposition 1.2 in Hatcher's notes). The fact that if $\mathscr{B}_Y$ is a basis for Y, then a function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for every $B \in \mathscr{B}_Y$ (see Theorem 2.9 in Crossley's book).

Major Examples: The basis $\{(a, b) : a < b\}$ for $\mathbb{R}$; the basis $\{[a, b) : a < b\}$ for $\mathbb{R}_l$ (a.k.a. $\mathbb{R}_h$); the basis $\{B_r(a) : a \in \mathbb{R}^n, r > 0\}$ for $\mathbb{R}^n$.

Interior, Closure, Boundary

Definitions: Interior, closure, boundary.

Notations: Let X be a topological space: the *closure* of $A \subset X$ is usually written as $\bar{A}$; the *interior* of $A \subset X$ is usually written as $\mathrm{int}(A)$; the *boundary* of $A \subset X$ is usually written as ∂A.

Major Facts/Results: Proposition 1.1 in Hatcher's notes.

Metric Spaces

Definitions: Metric on a set, metric ball $B_r(a)$ in X centered at $a \in X$ with radius $r > 0$, metric topology on X: topology generated by the basis $\mathscr{B} = \{\text{all metric balls in } X\}$.

Major Facts/Results: The "Sequence Lemma": Let A be a subset of a metric space X; for every $x \in \bar{A}$, there is a sequence $\{x_n\}$ in A that converges to x.

Major Examples: "Trivial" or "discrete" metric ($d(x, y) = 1$ if and only if $x \neq y$). Usual "round" metric on $\mathbb{R}^2$; the "square" metric on $\mathbb{R}^2$ given by the formula

$$d\big((x_1, x_2), (y_1, y_2)\big) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \ .$$

Hausdorff Spaces

Definitions: Hausdorff space.

Major Facts/Results: The limit of a sequence is unique in a Hausdorff space. Metric spaces are Hausdorff.

Major Examples: Topology on a finite set: determine whether or not such a space is Hausdorff. Line with two origins.

Connectedness

Definitions: Separation of a space, connected (sub)space, disconnected (sub)space.

Major Facts/Results: The closure of a connected subspace is connected. The image of a connected (sub)space under a continuous function is connected. Intermediate Value Theorem.

Major Examples: $\mathbb{R}_l$, $\mathbb{Q}$, $\mathbb{R} - \{0\}$, $\mathbb{R}^n - \{\text{origin}\}$ for $n > 1$, intervals in $\mathbb{R}$, the "Deleted Comb space": $\mathscr{C} = \{(0, 1)\} \cup \{(x, 0) : 0 \leq x \leq 1\} \cup \{(1/n, y) : n \in \mathbb{N}, 0 \leq y \leq 1\}$ with subspace topology from $\mathbb{R}^2$.

Path Connectedness

Definitions: Path from a point x to a point y in a space X, path connected space.

Major Facts/Results: Path-connectedness implies connectedness. The image of a path-connected (sub)space under a continuous function is path-connected.

Pasting Lemma: Let X and Y be spaces. Assume that $X = A \cup B$ for closed subspaces A and B. Suppose that $f : A \to Y$ and $g : B \to Y$ are continuous functions, and $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for all $x \in A$ and $h(x) = g(x)$ for all $x \in B$ is defined, and h is a continuous function. *In other words, if $f(x)$ and $g(x)$ are defined on closed sets A and B, respectively, and they agree on the overlap $A \cap B$, then f and g paste-together to define a continuous function $h(x)$ on $A \cup B$.*

Major Examples: $\mathbb{R}_l$, $\mathbb{Q}$, $\mathbb{R} - \{0\}$, $\mathbb{R}^n - \{\text{origin}\}$ for $n > 1$, intervals in $\mathbb{R}$, the "Deleted Comb space": $\mathscr{C} = \{(0,1)\} \cup \{(x,0) : 0 \leq x \leq 1\} \cup \{(1/n, y) : n \in \mathbb{N}, 0 \leq y \leq 1\}$ with subspace topology from $\mathbb{R}^2$.