MAT 145 : Midterm Solutions

Michael Williams

Each problem is worth 15 points.

1. Let X be the set $\{a, b, c, d\}$ with the topology $\mathcal{T}_X = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, and let Y be the (same) set $\{a, b, c, d\}$ with the topology $\mathcal{T}_Y = \{\varnothing, Y, \{b\}, \{d\}, \{b, d\}\}$. Consider the function $f : X \to Y$ defined by $f(a) = b$, $f(b) = d$, $f(c) = a$, and $f(d) = a$. Prove that $f : X \to Y$ is continuous.

   Solution: All we need to do is verify that the pre-image of every open set of Y is open in X. We calculate the relevant pre-images directly:

   - $f^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$ (is automatic).
   - $f^{-1}(Y) = X \in \mathcal{T}_X$ (is automatic).
   - $f^{-1}(\{b\}) = \{a\} \in \mathcal{T}_X$.
   - $f^{-1}(\{d\}) = \{b\} \in \mathcal{T}_X$.
   - $f^{-1}(\{b, d\}) = \{a, b\} \in \mathcal{T}_X$.

   We have shown that $f^{-1}(O) \in \mathcal{T}_X$ for every $O \in \mathcal{T}_Y$, so f is continuous.

2. Let X be a connected space, and let $Y = \{0\} \cup \{\, 1/n : n \in \mathbb{N}\}$ be considered as a subspace of $\mathbb{R}$ (with its usual topology). Prove that any continuous function $f : X \to Y$ is constant.

   Solution: We show that $f(x_1) = f(x_2)$ for every $x_1, x_2 \in X$; this will establish that $f(X)$ consists of a single point. We actually proceed by contradiction: suppose that some pair $x_1, x_2 \in X$ have different values under f, say, $f(x_1) < f(x_2)$. To simplify things, we extend the co-domain of f to $\mathbb{R}$; so $f : X \to \mathbb{R}$ is continuous with $f(X) \subset Y \subset \mathbb{R}$.

   One approach is to apply the Intermediate Value Theorem. For any $r \in \mathbb{R}$ with $f(x_1) < r < f(x_2)$, we can apply the Intermediate Value Theorem to conclude that $r \in f(X)$. Doing this for every $r \in \big(f(x_1), f(x_2)\big)$ establishes that

$\big[f(x_1), f(x_2)\big] \subset f(X)$. It is obvious that Y does not contain any intervals, so $f(X)$ cannot contain the interval $\big[f(x_1), f(x_2)\big]$; this gives a contradiction. Therefore, f must be a constant function.

Another approach is to use a basic *connectedness argument* as in the *proof of the Intermediate Value Theorem*. There exists a number $r \in \mathbb{R} - Y$ (an irrational number will suffice) such that $f(x_1) < r < f(x_2)$. Then $X = f^{-1}(-\infty, r) \cup f^{-1}(r, \infty)$ is a separation of X (by continuity of f and the containments $f(x_1) \in (-\infty, r)$ and $f(x_2) \in (r, \infty)$). This contradicts that X is connected. Therefore, f must be a constant function.

3. Let X be a Hausdorff space, and suppose that x_1, x_2, x_3 are three distinct points in X. Prove that there exist open sets V_1, V_2, V_3 each containing one, and only one, of the points x_1, x_2, x_3.

   Solution: We first construct disjoint neighborhoods for each pair of points in $\{x_1, x_2, x_3\}$ using the fact that X is a Hausdorff space:

   - There are disjoint neighborhoods U_1, U_2 of x_1, x_2 respectively.
   - There are disjoint neighborhoods U_1', U_3' of x_1, x_3 respectively.
   - There are disjoint neighborhoods U_2'', U_3'' of x_2, x_3 respectively.

   Now set $V_1 = U_1 \cap U_1'$, $V_2 = U_2 \cap U_2''$, and $V_3 = U_3' \cap U_3''$. We easily see that V_i is a neighborhood of x_i for $i = 1, 2, 3$, because finite intersections of open sets are open. Furthermore, $x_1 \notin U_2$ and $x_1 \notin U_3'$ implies that $x_1 \notin V_2$ and $x_1 \notin V_3$; $x_2 \notin U_1$ and $x_2 \notin U_3''$ implies that $x_2 \notin V_1$ and $x_2 \notin V_3$; $x_3 \notin U_1'$ and $x_3 \notin U_2''$ implies that $x_3 \notin V_1$ and $x_3 \notin V_2$. Therefore each open set V_i contains one, and only one, of the points x_1, x_2, x_3.

4. Consider the interval $[0, 1]$ with its usual topology. Give a self-contained proof that any continuous function $f : [0, 1] \to [0, 1]$ has a *fixed point*, that is, $f(x) = x$ for some $x \in [0, 1]$.

   You may use the fact that sums and differences of real-valued continuous functions are continuous.

   Solution: If $f(0) = 0$ or $f(1) = 1$, then there would be nothing to prove. So we may assume that $f(0) \neq 0$ and $f(1) \neq 1$. Since $f([0, 1]) \subset [0, 1]$, we deduce that $0 < f(0)$ and $f(1) < 1$. Define a function $g : [0, 1] \to \mathbb{R}$ by $g(x) = x - f(x)$.

The function g is the difference of two real-valued continuous functions, so g is continuous. Our previous observations about $f(0)$ and $f(1)$ allow us to conclude that $g(0) < 0 < g(1)$. We apply the Intermediate Value Theorem to the function $g(x)$: we conclude that there exists some $x \in [0,1]$ for which $g(x) = 0$; hence $f(x) = x$. This establishes that f has a fixed point.

5. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function. Prove that $f(\bar{A}) \subset \overline{f(A)}$ for any subset $A \subset X$.

   Solution: An element-chasing argument will suffice. Let $A \subset X$ be given, and suppose $x \in \bar{A}$; we want to show that $f(x) \in \overline{f(A)}$. Let O be a neighborhood of $f(x)$ in Y; we will show that $O \cap f(A) \neq \varnothing$. By continuity of f, the pre-image $f^{-1}(O)$ is a neighborhood of x in X. Since $x \in \bar{A}$, we must have $f^{-1}(O) \cap A \neq \varnothing$; so there exists an element $a \in f^{-1}(O) \cap A$. Thus $f(a) \in O \cap f(A)$; this follows by definitions of $f^{-1}(O)$ and $f(A)$. This shows that $O \cap f(A) \neq \varnothing$. Since O was an arbitrary neighborhood of $f(x)$, we conclude that $f(x) \in \overline{f(A)}$. This establishes that $f(\bar{A}) \subset \overline{f(A)}$.