MAT 145 : Final Exam Study Guide

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Remarks

The exam will mostly consists of proofs; there will also be some true/false questions. Your exam solutions are expected to be written clearly. You may apply theorems (without proof) if they were proved in class or in Crossley's book, Hatcher's notes, or the set theory notes. You may not use results proved on homework, quizzes, or on the midterm: it is possible that an exam question may actually be (part of) a homework problem, a quiz problem, or a midterm problem. If you are not sure about whether or not you can apply a result while taking the test, you may quietly ask me.

Practice Problems: All of the *problems for understanding* from Hw 1, 2, 3, 4, and 5 (except Problem 15 in Chapter 1 of Hatcher's notes and any problems dealing with quotient spaces), and all problems from Quizzes 1, 2, and 3.

Special Remark: *If your goal is to just pass the course with a C- grade*, you will want to focus on mastering the midterm exam material (**Set Theory (Basics) up to Path Connectedness**); this material will make a prominent appearance on the final exam. Demonstrating an A-level mastery of the midterm material will ensure that you pass the course.

Set Theory (Basics)

Definitions: **Sets**: elements, subsets, index set, union, intersection, complement, difference of sets, Cartesian product, element-chasing; **Functions**: domain (source), co-domain (target), image, pre-image (inverse image), injection, surjection, bijection, composition, inverse function, identity function.

Major Facts/Results: Distributive Laws and DeMorgan's Laws.

Topological Spaces and Continuous Functions (Basics)

Definitions: Topology (collection of "open sets") on a set (in terms of the axioms), "closed sets" with respect to a topology (and how to rephrase the axioms for a topology in terms of closed sets), indiscrete topology for any set X, discrete topology for any set X, topological space, neighborhood of a point in a topological space, continuous function from a topological space to another.

Major Facts/Results: If $f : X \to Y$ is a function from a space X to another space Y, then f is continuous if and only if the pre-image of closed sets are closed. Compositions of continuous functions are continuous.

Major Examples: Topologies on finite sets, usual topology on $\mathbb{R}^n$ (in terms of open balls of the form $B_r(a)$ where $a \in \mathbb{R}^n$ and $r > 0$), lower limit (also called half-open interval) topology on $\mathbb{R}$ (written as $\mathbb{R}_l$ or $\mathbb{R}_h$).

Subspace Topology (Basics)

Definitions: Subspace topology.

Major Facts/Results: The restriction of a continuous function to a subspace is continuous. If X is a space and Y is an open (resp. closed) subspace, then open (resp. closed) sets in Y are also considered open (resp. closed) in X.

Major Examples: Unit sphere $\mathbb{S}^n$ as a subspace of $\mathbb{R}^{n+1}$ (for $n = 0, 1, 2$), torus, cylinder (annulus), Möbius band.

Basis for a Topology

Definitions: Basis for a given topology, basis element, topology generated by a basis.

Major Facts/Results: Criterion for a collection $\mathscr{B} = \{B_\alpha\}$ of subsets of X to form a basis for a topology on X (see Proposition 1.2 in Hatcher's notes). The fact that if $\mathscr{B}_Y$ is a basis for Y, then a function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for every $B \in \mathscr{B}_Y$ (see Theorem 2.9 in Crossley's book).

Major Examples: The basis $\{(a, b) : a < b\}$ for $\mathbb{R}$; the basis $\{[a, b) : a < b\}$ for $\mathbb{R}_l$ (a.k.a. $\mathbb{R}_h$); the basis $\{B_r(a) : a \in \mathbb{R}^n, r > 0\}$ for $\mathbb{R}^n$.

Interior, Closure, Boundary

Definitions: Interior, closure, boundary.

Notations: Let X be a topological space: the *closure* of $A \subset X$ is usually written as $\bar{A}$; the *interior* of $A \subset X$ is usually written as $\text{int}(A)$; the *boundary* of $A \subset X$ is usually written as ∂A.

Major Facts/Results: Proposition 1.1 in Hatcher's notes.

Metric Spaces

Definitions: Metric on a set, metric ball $B_r(a)$ in X centered at $a \in X$ with radius $r > 0$, metric topology on X: topology generated by the basis $\mathscr{B} = \{\text{all metric balls in } X\}$.

Major Facts/Results: The "Sequence Lemma": Let A be a subset of a metric space X; for every $x \in \bar{A}$, there is a sequence $\{x_n\}$ in A that converges to x.

Major Examples: "Trivial" or "discrete" metric ($d(x, y) = 1$ if and only if $x \neq y$). Usual "round" metric on $\mathbb{R}^2$; the "square" metric on $\mathbb{R}^2$ given by the formula

$$d\big((x_1, x_2), (y_1, y_2)\big) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \ .$$

Hausdorff Spaces

Definitions: Hausdorff space.

Major Facts/Results: The limit of a sequence is unique in a Hausdorff space. Metric spaces are Hausdorff.

Major Examples: Topology on a finite set: determine whether or not such a space is Hausdorff. Line with two origins.

Connectedness

Definitions: Separation of a space, connected (sub)space, disconnected (sub)space.

Major Facts/Results: The closure of a connected subspace is connected. The image of a connected (sub)space under a continuous function is connected. Intermediate Value Theorem.

Major Examples: $\mathbb{R}_l$, $\mathbb{Q}$, $\mathbb{R}-\{0\}$, $\mathbb{R}^n-\{\text{origin}\}$ for $n>1$, intervals in $\mathbb{R}$.

Path Connectedness

Definitions: Path from a point x to a point y in a space X, path connected space.

Major Facts/Results: Path-connectedness implies connectedness. The image of a path-connected (sub)space under a continuous function is path-connected.

Pasting Lemma: Let X and Y be spaces. Assume that $X = A \cup B$ for closed subspaces A and B. Suppose that $f : A \to Y$ and $g : B \to Y$ are continuous functions, and $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for all $x \in A$ and $h(x) = g(x)$ for all $x \in B$ is defined, and h is a continuous function. *In other words, if $f(x)$ and $g(x)$ are defined on closed sets A and B, respectively, and they agree on the overlap $A \cap B$, then f and g paste-together to define a continuous function $h(x)$ on $A \cup B$.*

Major Examples: $\mathbb{R}_l$, $\mathbb{Q}$, $\mathbb{R}-\{0\}$, $\mathbb{R}^n-\{\text{origin}\}$ for $n>1$, intervals in $\mathbb{R}$.

Compactness

Definitions: Open covering, subcovering (refinement), compact space.

Major Results: A closed subspace of a compact space is necessarily compact. A compact subspace of a Hausdorff space is necessarily closed. The image of a compact space under a continuous function is compact. Extreme Value Theorem.

Major Examples: Closed interval in $\mathbb{R}$, $\mathbb{R}$ with the finite complement topology.

Homeomorphisms

Definitions: Homeomorphism, Homeomorphic spaces.

Notation: If X and Y are homeomorphic spaces, we write $X \cong Y$; if X and Y are not homeomorphic, we write $X \ncong Y$.

Major Examples: All open intervals (of finite length or infinite length) in $\mathbb{R}$ are homeomorphic.

All finite length closed intervals are homeomorphic. The closed unit disk in $\mathbb{R}^2$ is homeomorphic to the square $[0,1] \times [0,1] = [0,1]^2$; you do NOT have to know the proof.

Nonhomeomorphic spaces[1]: $\mathbb{S}^1 \not\cong \mathbb{R}$, $\mathbb{S}^1 \not\cong \mathbb{S}^2$, $\mathbb{S}^1 \not\cong [0,1]$, $\mathbb{S}^2 \not\cong \mathbb{R}^2$, $\mathbb{R} \not\cong \mathbb{R}_l = \mathbb{R}_h$, $\mathbb{Q} \not\cong \mathbb{Z}$.

Finite Products

Definitions: Product topology on $X \times Y$ and its standard basis, projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$.

Major Results: The projection maps (mentioned above) are continuous. A function $f : Z \to X \times Y$ is continuous if and only if the associated functions $(p_X \circ f) : Z \to X$ and $(p_Y \circ f) : Z \to Y$ are both continuous.

Let X and Y be spaces, and let $X \times Y$ denote the product space. Then

- X and Y are both Hausdorff if and only if $X \times Y$ is Hausdorff.
- X and Y are both connected if and only if $X \times Y$ is connected.
- X and Y are both compact if and only if $X \times Y$ is compact.

A subspace of $\mathbb{R}^n$ (with the usual topology) is compact if and only if it is closed and bounded (in the usual metric on $\mathbb{R}^n$).

Infinite Products

Definitions: Product topology and its standard basis, box topology and its standard basis.

Major Examples: $\mathbb{R}^\infty$ with the *product topology* (written as $\mathbb{R}^\infty_{product}$), $\mathbb{R}^\infty$ with the *box topology* (written as $\mathbb{R}^\infty_{box}$).

[1] All topologies here are the usual ones, except for $\mathbb{R}_l = \mathbb{R}_h$.