MAT 116 In-class Problems (#7)

July 20, 2010 and July 21, 2010

Let $\mathbb{Z}_+$ denote the set of *nonnegative* integers; if $n \in \mathbb{N}$, let $n\mathbb{Z}_+$ denote the *nonnegative* multiples of n. Given $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, let $\Sigma_{k,n}$ denote the set of all *nonnegative* integral solutions of the equation $e_1 + e_2 + \cdots + e_k = n$; in symbols, we have

$$\Sigma_{k,n} = \{(e_1, e_2, \ldots, e_k) \in \mathbb{Z}_+^k : e_1 + e_2 + \cdots + e_k = n\} \ .$$

We also adopt the convention that $C(n,k) = \binom{n}{k} = 0$ whenever $k > n$.

Example 1. Binomial coefficents: $\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{m}$ for a fixed $m \in \mathbb{N}$. The generating function is $g(x) = (1+x)^m$.

Example 2. Consider the sequence $h_0, h_1, h_2, \ldots$ where $h_n = n$. The generating function is $g(x) = \frac{x}{(1-x)^2}$.

Example 3. Consider the sequence $h_0, h_1, h_2, \ldots$ where $h_n = n^2$. The generating function is $g(x) = \frac{x(x+1)}{(1-x)^3}$.

Example 4. For a fixed $k \in \mathbb{N}$, consider the sequence $h_0, h_1, h_2, \ldots$ where $h_n = |\Sigma_{k,n}| = \binom{n+k-1}{k-1}$. The generating function is $g(x) = \frac{1}{(1-x)^k}$.

Example 5. Consider the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$. For each $n \in \{0, 1, 2, \ldots, 12\}$, let $h_n = |\{n\text{–combinations of } S\}|$. The generating function for the sequence $h_0, h_1, h_2, \ldots, h_{12}$ is

$$g(x) = 1 + 3x + 6x^2 + 10x^3 + 14x^4 + 17x^5 + 18x^6 + 17x^7 + 14x^8 + 10x^9 + 6x^{10} + 3x^{11} + x^{12} \ .$$

Example 6. Consider the sequence $h_0, h_1, h_2, \ldots$ where

$$h_n = |\{(e_1, e_2, e_3, e_4) \in \Sigma_{4,n} : e_1 \text{ is even}, e_2 \text{ is odd}, e_3 \leq 4, 1 \leq e_4\}| \ .$$

The generating function is $g(x) = \frac{x^2(1-x^5)}{(1-x^2)^2(1-x)^2}$.

Example 7. Consider the sequence $h_0, h_1, h_2, \ldots$ where

$$h_n = |\{(e_1, e_2, e_3, e_4) \in \Sigma_{4,n} : e_1 \in 3\mathbb{Z}_+, e_2 \in 4\mathbb{Z}_+, e_3 \in 2\mathbb{Z}_+, e_4 \in 5\mathbb{Z}_+\}| \ .$$

The generating function is $g(x) = \frac{1}{(1-x^3)(1-x^4)(1-x^2)(1-x^5)}$.